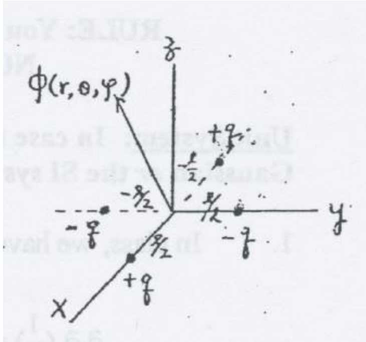
Sample: Electrodynamics - Electric Potential

Problem 2: Consider the distribution of point charges as shown in the diagram. Show that:



(a) the quadruple moment tensor is given by the following:

$$Q = \frac{3}{2}\begin{pmatrix} ql^2 & 0 & 0 \\ 0 & -ql^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution: in the Cartesian coordinate system the coordinates of all the point charges are given by $q_1(+q) \to \left(\frac{l}{2}, 0{,}0\right), q_2(-q) \to \left(0, -\frac{l}{2}, 0\right), q_3(+q) \to \left(-\frac{l}{2}, 0, 0\right), q_4(-q) \to \left(0, \frac{l}{2}, 0\right)$. Each of them is at the distance $r = \frac{l}{2}$ from the origin. The quadruple moment tensor is symmetric and its trace is equal to zero. That implies that we have to calculate only 5 components of this tensor. The components Q_{ij} of the quadruple moment tensor are defined by

$$Q_{ij} = \sum_{k=1}^{N} q_k (3x_{ik}x_{jk} - \delta_{ij} r_k^2),$$

Where x_{ik} is the $x_i (i = 2 \to x_2 = y)$ coordinate of k –th charge. Let us calculate now the components of this matrix:

1) $Q_{xx} = Q_{11} = \sum_{k=1}^{N} q_k (3x_k x_k - r_k^2) = +q\left(\frac{3l^2}{4} - \frac{l^2}{4}\right) - q\left(0 - \frac{l^2}{4}\right) + q\left(\frac{3l^2}{4} - \frac{l^2}{4}\right) - q\left(0 - \frac{l^2}{4}\right) = \frac{3}{2} q l^2.$
2) $Q_{zz} = Q_{33} = \sum_{k=1}^{N} q_k (3z_k z_k - r_k^2) = -\sum_{k=1}^{N} q_k \cdot r_k^2$ $(\text{for each charge z} - \text{compoent is } 0) = -\frac{l^2}{4}(+q - q + q - q) = 0.$
3) $Q_{xx} + Q_{yy} + Q_{zz} = 0 \implies Q_{yy} = Q_{22} = -Q_{xx} - Q_{zz} = -\frac{3}{2} q l^2.$
4) $Q_{xy} = Q_{yx} = Q_{12} = \sum_{k=1}^{N} q_k 3x_k y_k = +3q\left(\frac{l}{2} \cdot 0\right) - 3q\left(-\frac{l}{2} \cdot 0\right) + 3q\left(-\frac{l}{2} \cdot 0\right) - 3q\left(0 \cdot \frac{l}{2}\right) = 0.$
5) $Q_{xz} = Q_{zx} = Q_{13} = \sum_{k=1}^{N} q_k 3x_k z_k = \sum_{k=1}^{N} q_k (3x_k \cdot 0) = 0.$
6) $Q_{yz} = Q_{zy} = Q_{23} = \sum_{k=1}^{N} q_k 3y_k z_k = \sum_{k=1}^{N} q_k (3y_k \cdot 0) = 0.$

From this data we can reconstruct the quadruple moment tensor:

$$Q_{ij} = \begin{pmatrix} \frac{3}{2} q l^2 & 0 & 0 \\ 0 & -\frac{3}{2} q l^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} ql^2 & 0 & 0 \\ 0 & -ql^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Answer: the quadruple moment tensor is

$$Q_{ij} = \frac{3}{2}\begin{pmatrix} ql^2 & 0 & 0 \\ 0 & -ql^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) The quadruple potential is given (in spherical coordinates) by:

$$\Phi^{(4)} = \frac{3}{4} k_E q l^2 \cdot \frac{\sin^2\theta \, \cos 2\phi}{r^3}.$$

Solution: the quadruple potential can be calculated from formula

$$\Phi^{(4)}(x,y,z) = \sum_{i,j=1}^{3} \frac{k_E Q_{ij} x_i x_j}{2r^5},$$

Where Q_{ij} is the quadruple moment tensor and x_j is the j —th component of the radius vector in the Cartesian system. Since there are only two nonzero components of Q_{ij}, the potential takes form

$$\Phi^{(4)}(x,y,z) = \sum_{i,j=1}^{3} \frac{k_E Q_{ij} x_i x_j}{2r^5} = \frac{k_E (Q_{11}x^2 + Q_{22}y^2)}{2r^5} = \frac{3k_E q l^2 (x^2 - y^2)}{4r^5}$$

Let us now transform the Cartesian coordinates into spherical coordinates:

$$\Phi^{(4)}(r,\theta,\phi) = \frac{3k_E q l^2((r\sin\theta\cos\phi)^2 - (r\sin\theta\sin\phi)^2)}{4r^5}$$
$$= \frac{3k_E q l^2 \sin^2\theta\,(\cos^2\phi - \sin^2\phi)}{4r^3} = \frac{3}{4}k_E q l^2 \cdot \frac{\sin^2\theta\ \cos 2\phi}{r^3}.$$

So finally we obtain the quadruple potential (in spherical coordinates):

$$\Phi^{(4)} = \frac{3}{4}k_E q l^2 \cdot \frac{\sin^2\theta\ \cos 2\phi}{r^3}.$$

Answer: the quadruple potential is given (in spherical system) by:

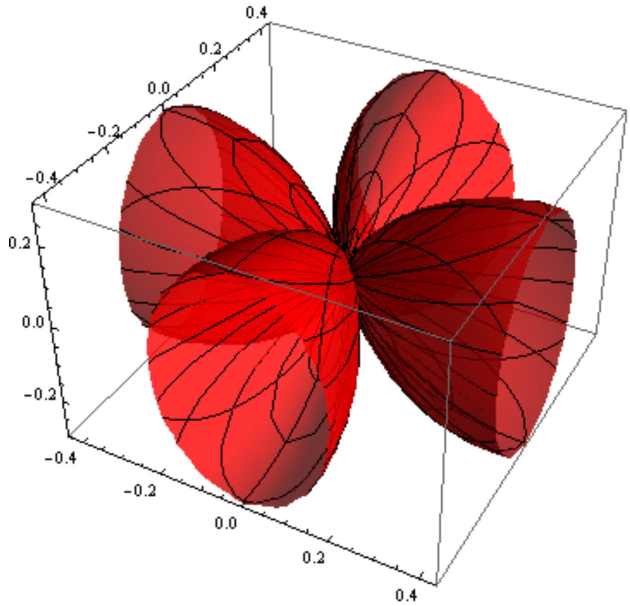
$$\Phi^{(4)}(r,\theta,\phi) = \frac{3}{4}k_E q l^2 \cdot \frac{\sin^2\theta\ \cos 2\phi}{r^3}.$$

(c) Sketch the potential as a function of angles θ and ϕ for a fixed r.

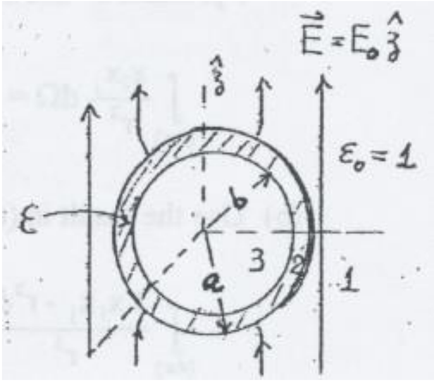
Solution: to sketch the potential $\Phi^{(4)}$ for fixed r it is convenient to introduce the dimensionless potential

$$\phi(\theta,\phi) = \frac{r^3}{k_E q l^2} \cdot \Phi^{(4)}(\theta,\phi) = \frac{3}{4}\sin^2\theta \cdot \cos 2\phi$$

We have plotted the potential $\phi(\theta,\phi)$ using Mathematica (Code: $\mathrm{SphericalPlot3D}[3/4 * (\mathrm{Sin}[\theta])^\wedge 2 * \mathrm{Cos}[2*\phi], \theta, \phi, \mathrm{PlotStyle} \to \mathrm{Directive}[\mathrm{Red}, \mathrm{Opacity}[0.8]]]$):



Problem 3: a hollow dielectric sphere (of l.i.h. material of dielectric constant ϵ and radii a and $b < a$) is in a uniform external filed $\vec{E}$ in vacuum as shown. Try to justify the following forms for the potentials in three regions:



$$V_1 = -E_0 \cos\theta \left(r - \frac{A}{r^2}\right),$$

$$V_2 = -E_0 \cos\theta \left(Br - \frac{C}{r^2}\right),$$

$$V_3 = -E_0 D \cos\theta\, r,$$

and use them to show that electric field in the cavity ($\vec{E_3}$) is uniform and is given by following expression:

$$\vec{E_3} = \frac{9\epsilon}{\left((\epsilon+2)(2\epsilon+1) - 2(\epsilon-1)^2\left(\frac{b}{a}\right)^3\right)}\vec{E} \text{ (in Gaussian units).}$$

Solution: we will provide all the calculations in the Gaussian units.

1) In all three regions the potentials V_1, V_2, V_3satisfy the Laplace equation $\Delta V = 0$. Our system has azimuthal symmetry (this means that solutions does not depend on the angle ϕ). In this case the general solution of equation $\Delta V = 0$ has the form

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$P_l(\cos\theta)$ are Legendre polynomials in the variable of $\cos\theta$: $P_0(x) = 1, P_1(x) = x$ ...
Let us consider the potentials V_1, V_2, V_3more specifically:

a) The external field is $\vec{E} = E_0 \cdot \hat{z}$, so the potential that corresponds to this field is

$$\vec{E} = -\nabla V \implies V = -\int(\vec{E}\cdot d\vec{r}) = -E_0\int dz = -E_0 z$$

In the spherical system this potential has the form $V = -E_0 r\cos\theta$. For large r ($r \gg a$) the potential V_3 should look like the potential $V = -E_0 r \cos\theta$. For large r the value of $\frac{B_l}{r^{l+1}}$ is negligible small, therefore

$$\sum_{l=0}^{\infty} A_l r^l \cdot P_l(\cos\theta) = -E_0 r\cos\theta$$

$$A_0 + A_1 r\cos\theta + \sum_{l=2}^{\infty} A_l r^l \cdot P_l(\cos\theta) = -E_0 r\cos\theta$$

The Legendre polynomials form the linear independent system of functions, so this condition can be satisfied only when $l = 1$ and

$$A_l = 0 \; (l \neq 1)$$
$$A_1 = -E_0 \; (l = 1)$$

The potential $V_3$the takes the form

$$V_3(r,\theta) = (A_1 r + \frac{B_1}{r^2})\cdot P_1(\cos\theta) = \left(-E_0 r + \frac{B_1}{r^2}\right)\cos\theta = -E_0\cos\theta\left(r - \frac{A}{r^2}\right),$$

Where $A = \frac{B_1}{E_0}$.

b) The potential V is a continuous function on the boundary. That implies

$$V_2(a) = V_1(a).$$

$$V_2(r,\theta) = \sum_{l=0}^{\infty}\left(B_l r^l + \frac{C_l}{r^{l+1}}\right)P_l(\cos\theta)$$

From this boundary condition we get

$$\sum_{l=0}^{\infty}\left(B_l a^l + \frac{C_l}{a^{l+1}}\right)P_l(\cos\theta) = -\,E_0\cos\theta\left(a - \frac{A}{a^2}\right)$$

Again, since Legendre polynomials form the linear independent system of functions, this equality can be possible only if $l = 1$ and

$$B_l = 0\ (l \neq 1)$$

$$C_l = 0\ (l \neq 1).$$

From these conditions we obtain the look of the potential V_2:

$$V_2(r,\theta) = \left(B_1 r + \frac{C_1}{r^2}\right)\cos\theta = V_2 = -E_0\cos\theta\left(Br - \frac{C}{r^2}\right),$$

Where $B_1 = -E_0 B$ and $C_1 = E_0 C$.

c) The potential in third region has the form

$$V_1(r,\theta) = \sum_{l=0}^{\infty}\left(D_l r^l + \frac{E_l}{r^{l+1}}\right)P_l(\cos\theta)$$

In this case $E_l = 0$ for all l —otherwise the potential will blow up at the origin. Thus

$$V_1(r,\theta) = \sum_{l=0}^{\infty} D_l r^l P_l(\cos\theta)$$

Using boundary condition $V_3(b) = V_2(b)$ we obtain:

$$\sum_{l=0}^{\infty} D_l b^l P_l(\cos\theta) = D_0 + D_1 b\cos\theta + \sum_{l=2}^{\infty} D_l b^l P_l(\cos\theta) = -E_0\cos\theta\left(Bb - \frac{C}{b^2}\right),$$

From linear independency of $P_l(\cos\theta)$ we once again get: $l = 1, D_l = 0\ (l \neq 1)$.

$$V_1(r,\theta) = D_1 r\cos\theta = -E_0 D\, r\cos\theta\ (D_1 = -E_0 D)$$

Answer: the potentials in three regions are given by:

$$V_1 = -E_0\cos\theta\left(r - \frac{A}{r^2}\right),$$

$$V_2 = -E_0\cos\theta\left(Br - \frac{C}{r^2}\right),$$

$$V_3 = -E_0 D\cos\theta\, r.$$

2) In the expressions for the potentials appear four constants A, B, C, D that can be determined using boundary conditions. The electric field in the cavity is

$$\overrightarrow{E_3} = -\nabla V_3 = -\nabla(-E_0 D\cos\theta\, r) = E_0 \mathrm{D}\nabla z = E_0 D\hat{z} = D \cdot \vec{E}.$$

From this formula we can see that field in the cavity is constant and has the same direction as the external field. To determine this field fully we should find the unknown coefficient D. We will find D using the boundary conditions. The potential itself is a continuous function on the boundary. This condition implies

$$V_3(b) = V_2(b),\ \ V_2(a) = V_1(a).$$

The second boundary condition has the deal with the first derivative of the potential on the boundary. Since there are no free charges on the surfaces of the hollow sphere, it takes form

$$\frac{\partial V_3(b)}{\partial r} = \epsilon\frac{\partial V_2(b)}{\partial r},\ \ \frac{\partial V_1(a)}{\partial r} = \epsilon\frac{\partial V_2(a)}{\partial r}.$$

From first two conditions we obtain:

$$-E_0 D\cos\theta\, b = -E_0\cos\theta\left(Bb - \frac{C}{b^2}\right) \implies \frac{C}{b^3} = B - D;$$

$$-E_0\cos\theta\left(a - \frac{A}{a^2}\right) = -\,E_0\cos\theta\left(Ba - \frac{C}{a^2}\right) \implies B - 1 = \frac{C - A}{a^3}.$$

From second one we get two additional relationships between constants A, B, C, D:

$$-E_0 D\cos\theta = -\epsilon\, E_0\cos\theta\left(B + \frac{2C}{b^3}\right) \implies \frac{D}{\epsilon} = B + \frac{2C}{b^3};$$

$$-E_0\cos\theta\left(1 + \frac{2A}{a^3}\right) = -\epsilon E_0\cos\theta\left(B + \frac{2C}{a^3}\right) \implies 1 + \frac{2A}{a^3} = \epsilon B + \frac{2\epsilon C}{a^3}.$$

From the system of equations

$$\frac{C}{b^3} = B - D,\ \ B - 1 = \frac{C - A}{a^3},\ \ \frac{D}{\epsilon} = B + \frac{2C}{b^3},\ \ 1 + \frac{2A}{a^3} = \epsilon B + \frac{2\epsilon C}{a^3}$$

We can determine the coefficient D. Using first and third equation we can express B as a function of D:

$$B = \frac{1 + 2\epsilon}{3\epsilon}\cdot D.$$

From first equation we can obtain now C as a function of D:

$$C = \frac{1 - \epsilon}{3\epsilon}b^2\cdot D.$$

Substituting B, C in the second equation we get A as a function of D:

$$A = a^3\left(1 - \frac{(1 + 2\epsilon) + (\epsilon - 1)\left(\frac{b}{a}\right)^3}{3\epsilon}\cdot D\right).$$

Substituting A, B, C in the fourth equation we obtain the value of D:

$$1 + \frac{2A}{a^3} = \epsilon B + \frac{2\epsilon C}{a^3}$$

$$3 - \frac{2D}{3\epsilon}\left((1 + 2\epsilon) + (\epsilon - 1)\left(\frac{b}{a}\right)^3\right) = \frac{1 + 2\epsilon}{3}\cdot D + \frac{2(1 - \epsilon)}{3}\left(\frac{b}{a}\right)^3\cdot D$$

$$3 = \frac{D}{3\epsilon}\left((1 + 2\epsilon)(\epsilon + 2) - 2\left(\frac{b}{a}\right)^3(\epsilon - 1)^2\right)$$

$$D = \frac{9\epsilon}{(1 + 2\epsilon)(\epsilon + 2) - 2\left(\frac{b}{a}\right)^3(\epsilon - 1)^2}.$$

Finally the field in the cavity is

$$\overrightarrow{E_3} = \frac{9\epsilon}{(1+2\epsilon)(\epsilon+2) - 2\left(\frac{b}{a}\right)^3 (\epsilon-1)^2} \cdot \vec{E}.$$

Answer: field in the cavity is $\overrightarrow{E_3} = \frac{9\epsilon}{(1+2\epsilon)(\epsilon+2)-2\left(\frac{b}{a}\right)^3(\epsilon-1)^2} \cdot \vec{E}$, where $\vec{E} = E_0 \cdot \hat{z}$ is the uniform external field.