Question.

Given the number 123456789. Is it possible to find a permutation (i.e. rearrangement that preserves the count of each number) such that the left most digit is evenly divisible by 1, the two left most digits are evenly divisible by 2, the three left most digits are divisibly by 3 and so on?

Solution.

We have number <d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8 d_9>, where digits form the permutations of 123456789. We need to find, what position will have every digit to qualify the answer.

1) As <d_1 d_2> should be divisible by **2**, <d_1 d_2 d_3 d_4> should be divisible by **4** (and it should be also divisible by **2**) and so on, all even digits (d_2, d_4, d_6, d_8) will be even (2, 4, 6, 8).

2) As <d_1 d_2 d_3 d_4 d_5> should be divisible by **5**, the last digit should be 0 or 5. As 0 is not used, so, d_5 = **5**.

3) <d_1 d_2 d_3 d_4> should be divisible by **4**. The divisibility rule for **4** states that if the last 2 digits are divisible by 4, then the whole number is divisible by 4. So <d_3 d_4> is divisible by **4**, d_3 is odd and d_4 is even, and d_3 is not equal to **5** (by item (2)). We can enumerate all numbers witch match this pattern: 12, 16, 32, 36, 72, 76, 92, 96. <d_3 d_4> is one of them.
We can notice, that d_4 by this pattern will be equal to 2 or to 6.

4) Using pattern from item (3) we can analyze digits <d_7 d_8>. As <d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8> should be divisible by **8**, this number will also be divisible by **4**. So the last two digits <d_7 d_8> will match the same rule as <d_3 d_4>, so d_8 will be also equal to 2 or to 6.

5) As it was proved in (3) and (4), d_4 and d_8 will be equal to 2 or to 6. In this case, d_2 and d_6 will be equal to 4 or to 8.

As we have found, d_5 is equal to **5**, d_4 and d_8 are equal to **2** or to **6**, d_2 and d_4 are equal to **4** or to **8**. This information will be used for further conclusions.

6) <d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8> should be divisible by **8**. The divisibility rule for **8** states, that last 3 digits should be divisible by 8, so <d_6 d_7 d_8> should be divisible by **8**. As we know, d_6 is equal to 4 or to 8, d_8 is equal to 2 or to 6 and d_7 is not equal to 5. We can match only a few numbers for this pattern: 416, 432, 472, 496, 816, 832, 872, 896. So <d_6 d_7 d_8> is one of them.

7) <d_1 d_2 d_3> should be divisible by **3**. The divisibility rule for **3** states, that the sum of all digits of number should be divisible by 3. Also we know, that d_1 and d_3 are odd and not equal to 5, d_2 is even and is equal to 4 or to 8. Only few numbers matches this pattern: 147, 183, 189, 381, 387, 741, 783, 981. So <d_1 d_2 d_3> is one of them.

8) <d_1 d_2 d_3 d_4 d_5 d_6> should be divisible by **6**, so it should be divisible by **3**, so the rule (7) can be applicable in this case. As sum of d_1, d_2 and d_3 is divisible by **3** (from (7)), so sum of d_4, d_5 and d_6 should be divisible by **3**. As we know, d_5 is equal to 5, d_6 is equal to 4 or 8 and d_4 is equal to 2 or to 6. There are only two numbers, which match this pattern: 258 and 654.

Now we can concatenate all patterns from (1) – (8).

1. If $<d_4 d_5 d_6> = 258$ (from (8)), then $<d_6 d_7 d_8>$ can be equal to 816 or to 896 (from (6)). So we can have $<d_1 d_2 d_3\ 2\ 5\ 8\ 1\ 6\ d_9>$ or $<d_1 d_2 d_3\ 2\ 5\ 8\ 9\ 6\ d_9>$.
In the first case we can not match $<d_1 d_2 d_3>$ to any of variants from (7), as 1 and 8 are already used. But for the second case we can match $<d_1 d_2 d_3>$ with 2 variants from (7): 147 and 741. And after matching the last digit we have got two variants of number:

d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
1	4	7	2	5	8	9	6	3
7	4	1	2	5	8	9	6	3

2. If $<d_4 d_5 d_6> = 654$ (from (8)), then $<d_6 d_7 d_8>$ can be equal to 472 or 432 (from (6)). So we can have $<d_1 d_2 d_3\ 6\ 5\ 4\ 7\ 2\ d_9>$ or $<d_1 d_2 d_3\ 6\ 5\ 4\ 3\ 2\ d_9>$.
In the first case we can match $<d_1 d_2 d_3>$ with next variants from (7): 183, 189, 381, 981. In the second case we can match $<d_1 d_2 d_3>$ with the next variants from (7): 189 and 981. And after matching the last digit we have got another 6 options:

d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
1	8	3	6	5	4	7	2	9
1	8	9	6	5	4	7	2	3
3	8	1	6	5	4	7	2	9
9	8	1	6	5	4	7	2	3
1	8	9	6	5	4	3	2	7
9	8	1	6	5	4	3	2	7

By default all matching numbers are divisible by **9**, as according to the divisibility rule for 9, the sum of all digits is divisible by 9.
The last thing to check is divisibility by **7** of number $<d_1 d_2 d_3 d_4 d_5 d_6 d_7>$. It is simply checked for the all 8 numbers found in solution. And we can found, that there is the only number, which matches the question and it is **3 8 1 6 5 4 7 2 9**.

Answer.

Such number exists. It is **381654729**.

Note.

Such number can be found by the bust of all permutations of 123456789 with checks for divisibility on each step. But the total number of all permutations is equal to 9! = 9*8*7*6*5*4*3*2*1 = 362880 which (including all divisibility checks) will take a noticeable large amount of time.

Divisibility rules see in:
https://en.wikipedia.org/wiki/Divisibility_rule