Answer on **Question** #61799 Programming & Computer Science

Let $x_0, x_1, \dots, x_n$ be $n+1$ distinct points on the real line and let $f(x)$ be a real-valued function defined on some interval $I = [a, b]$ containing these points. Then, there exists exactly one polynomial $P_n(x)$ of degree n, which matches $f(x)$ at $x_0, x_1, \dots, x_n$. This one polynomial named an interpolation polynomial.

Solution

Let the polynomial has the form $P_n(x) = \sum_{k=0}^{n} a_k x^k$ and satisfying conditions of interpolation $\forall i = 0..n$: $P_n(x_i) = f(x_i) = f_i$. Then we have a system of n linear equations with n unknown coefficients $\{a_k\}$:

$$\begin{bmatrix} 1 & \cdots & {x_0}^n \\ \vdots & \ddots & \vdots \\ 1 & \cdots & {x_n}^n \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ \vdots \\ f_n \end{bmatrix}$$

This system has exactly one solution $\{a_k\}$, because its matrix is nonsingular Vandermonde matrix and therefore there exists exactly one polynomial $P_n(x)$ of degree n, which matches $f(x)$ at $x_0, x_1, \dots, x_n$.