Answer on Question #85412, Physics / Other

Show that for two scalar fields f and g: $\nabla \cdot [\nabla f \ \times (f \ \nabla g)] = \ 0$.

Solution:

Using definitions of the vector differential operations,

$$\nabla \ \cdot \mathbf{A} = \nabla_i A_i = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla f = \boldsymbol{e_i} \nabla_i f = \boldsymbol{e_x} \frac{\partial f}{\partial x} + \boldsymbol{e_y} \frac{\partial f}{\partial y} + \boldsymbol{e_z} \frac{\partial f}{\partial z}$$

$$[\mathbf{A} \times \mathbf{B}] = e_i \varepsilon_{ijk} A_j B_k = \begin{vmatrix} \boldsymbol{e_x} & \boldsymbol{e_y} & \boldsymbol{e_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

we obtain

$$[\nabla f \ \times (f \ \nabla g)] = f \begin{vmatrix} \boldsymbol{e_x} & \boldsymbol{e_y} & \boldsymbol{e_z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix}$$

$$= f \left[\boldsymbol{e_x} \left(\frac{\partial f}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial y} \right) + \boldsymbol{e_y} \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial z} \right) + \boldsymbol{e_z} \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) \right]$$

Thus

$$\nabla \cdot [\nabla f \ \times (f \ \nabla g)] = \left[\frac{\partial f}{\partial x} \left(\frac{\partial f}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial y} \right) + \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial z} \right) + \frac{\partial f}{\partial z} \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) \right]$$

$$+ f \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) \right]$$

$$= \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial x} \frac{\partial f}{\partial z} \frac{\partial g}{\partial y} + \frac{\partial f}{\partial y} \frac{\partial f}{\partial z} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \frac{\partial f}{\partial x} \frac{\partial g}{\partial z} + \frac{\partial f}{\partial z} \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial z} \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}$$

$$+ f \left[\frac{\partial^2 f}{\partial x \partial y} \frac{\partial g}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial^2 g}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial z} \frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial z} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial x} \frac{\partial^2 g}{\partial z \partial y} \right.$$
$$\left. + \frac{\partial^2 f}{\partial x \partial z} \frac{\partial g}{\partial y} + \frac{\partial f}{\partial x} \frac{\partial^2 g}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \frac{\partial^2 g}{\partial x \partial z} \right] = 0 + f \times 0 = 0$$

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