Answer on Question #84387 – Physics – Quantum Mechanics

Find the probability distributions of the orbital angular momentum variables L2 and Lz for the following orbital state functions:

(a) Ψ(x) = f(r) sin θ cos ϕ,

(b) Ψ(x) = f(r)(cos θ)2,

(c) Ψ(x) = f(r) sin θ cos θ sin ϕ.

Here r, θ, ϕ are the usual spherical coordinates, and f(r) is an arbitrary radial function (not necessarily the same in each case) into which the normalization constant has been absorbed

Solution. Distributions are as follows:

$L_z\,\Psi(x)=-i\frac{h}{2\pi}\frac{\partial}{\partial\varphi}\psi$; $L^2\,\Psi(x)=-\frac{h^2}{4\pi^2}[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial}{\partial\varphi^2}]\psi$.

Then:

a) $L_z\,\Psi(x)=-i\frac{h}{2\pi}\frac{\partial}{\partial\varphi}\psi=-i\frac{h}{2\pi}\frac{\partial f(r)}{\partial\varphi}\sin\theta\cos\phi+i\frac{h}{2\pi}f(r)\sin\theta\sin\phi=-i\frac{h}{2\pi}\sin\theta(\frac{\partial f(r)}{\partial\varphi}\cos\phi-f(r)\sin\phi)$;

$L^2\,\Psi(x)=-\frac{h^2}{4\pi^2}[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial}{\partial\varphi^2}]\psi=-\frac{h^2}{4\pi^2}[\frac{1}{sin\theta}(\frac{\partial^2 f(r)}{\partial\theta^2}sin^2\theta cos\varphi+3\sin\theta\cos\theta\cos\phi\frac{\partial f(r)}{\partial\theta}+$

$+\cos 2\theta\cos\phi f(r))+\frac{1}{sin^2\theta}(\frac{\partial^2 f(r)}{\partial\varphi^2}sin\theta cos\varphi-2\frac{\partial f(r)}{\partial\varphi}sin\theta sin\varphi+f(r)sin\theta cos\varphi)]=$

$=-\frac{h^2}{4\pi^2}[\frac{\partial^2 f(r)}{\partial\theta^2}sin\theta cos\varphi+3\cos\theta\cos\phi\frac{\partial f(r)}{\partial\theta}+\frac{cos2\theta}{sin\theta}\cos\phi f(r)+\frac{\partial^2 f(r)}{\partial\varphi^2}\frac{cos\varphi}{sin\theta}-2\frac{\partial f(r)}{\partial\varphi}\frac{sin\varphi}{sin\theta}+f(r)\frac{cos\varphi}{sin\theta}]$;

b) $L_z\,\Psi(x)=-i\frac{h}{2\pi}\frac{\partial}{\partial\varphi}\psi=-i\frac{h}{2\pi}\frac{\partial f(r)}{\partial\varphi}(\cos\theta)^2$;

$L^2\Psi(x)=-\frac{h^2}{4\pi^2}[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial}{\partial\varphi^2}]\psi=-\frac{h^2}{4\pi^2}[\frac{\partial^2 f(r)}{\partial\theta^2}cos^2\theta-8cos\theta sin\theta\frac{\partial f(r)}{\partial\theta}+$

$+\frac{\partial f(r)}{\partial\theta}\frac{cos^3\theta}{sin\theta}-4\sin\theta\cos^2\theta f(r)+2\sin^2\theta f(r)+\frac{\partial^2 f(r)}{\partial\varphi^2}cos^2\theta]$;

c) $L_z\,\Psi(x)=-i\frac{h}{2\pi}\frac{\partial}{\partial\varphi}\psi=-i\frac{h}{2\pi}(\frac{\partial f(r)}{\partial\varphi}\sin\theta\cos\theta\sin\phi+f(r)\sin\theta\cos\theta\cos\phi)$;

$L^2\,\Psi(x)=-\frac{h^2}{4\pi^2}[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial}{\partial\varphi^2}]\psi=-\frac{h^2}{4\pi^2}[\frac{\partial^2 f(r)}{\partial\theta^2}sin\theta cos\theta sin\varphi+3\frac{\partial f(r)}{\partial\theta}cos^2\theta sin\varphi-$

$-2\frac{\partial f(r)}{\partial\theta}sin\varphi sin^2\theta+f(r)sin\varphi\frac{cos^3\theta}{sin\theta}-5cos\theta sin\theta sin\varphi f(r)+\frac{\partial^2 f(r)}{\partial\varphi^2}sin\theta cos\theta sin\varphi+$

$+2\frac{\partial f(r)}{\partial\varphi}sin\theta cos\theta cos\varphi-f(r)\sin\theta\cos\theta\sin\varphi]$.

Answer: a) $L_z\,\Psi(x)=-i\frac{h}{2\pi}\sin\theta(\frac{\partial f(r)}{\partial\varphi}\cos\phi-f(r)\sin\phi)$; $L^2\,\Psi(x)==-\frac{h^2}{4\pi^2}[\frac{\partial^2 f(r)}{\partial\theta^2}sin\theta cos\varphi+3\cos\theta\cos\phi\frac{\partial f(r)}{\partial\theta}+\frac{cos2\theta}{sin\theta}\cos\phi f(r)+\frac{\partial^2 f(r)}{\partial\varphi^2}\frac{cos\varphi}{sin\theta}-2\frac{\partial f(r)}{\partial\varphi}\frac{sin\varphi}{sin\theta}+f(r)\frac{cos\varphi}{sin\theta}]$;

b) $L_z\,\Psi(x)=-i\frac{h}{2\pi}\frac{\partial f(r)}{\partial\varphi}(\cos\theta)^2$; $L^2\Psi(x)=-\frac{h^2}{4\pi^2}[\frac{\partial^2 f(r)}{\partial\theta^2}cos^2\theta-8cos\theta sin\theta\frac{\partial f(r)}{\partial\theta}++\frac{\partial f(r)}{\partial\theta}\frac{cos^3\theta}{sin\theta}-4\sin\theta\cos^2\theta f(r)+2\sin^2\theta f(r)+\frac{\partial^2 f(r)}{\partial\varphi^2}cos^2\theta]$;

c) $L_z\,\Psi(x)=-i\frac{h}{2\pi}(\frac{\partial f(r)}{\partial\varphi}\sin\theta\cos\theta\sin\phi+f(r)\sin\theta\cos\theta\cos\phi)$; $L^2\,\Psi(x)=-\frac{h^2}{4\pi^2}[\frac{\partial^2 f(r)}{\partial\theta^2}sin\theta cos\theta sin\varphi+3\frac{\partial f(r)}{\partial\theta}cos^2\theta sin\varphi--2\frac{\partial f(r)}{\partial\theta}sin\varphi sin^2\theta+f(r)sin\varphi\frac{cos^3\theta}{sin\theta}-5cos\theta sin\theta sin\varphi f(r)+\frac{\partial^2 f(r)}{\partial\varphi^2}sin\theta cos\theta sin\varphi++2\frac{\partial f(r)}{\partial\varphi}sin\theta cos\theta cos\varphi-f(r)\sin\theta\cos\theta\sin\varphi]$.