Answer on Question #82306 - Physics - Mechanics – Relativity

Find the expression for the volume of liquid passing per second, V, through a pipe when the flow is steady. Assume the V proportional to (i) the coefficient of viscosity η of the liquid (ii) the radius r of the pipe and (iii) the pressure gradient, (ρ/l), causing the flow, where ρ is the pressure difference between the ends of the pipe and l is the length.

Solution

According to what is given, take a small piece of a tube of length $\mathrm{d}x$. On the sides of the tube the force $\mathrm{d}F$ acts:

$$\mathrm{d}F = \eta \cdot 2\pi r \cdot l \cdot \frac{\mathrm{d}v}{\mathrm{d}r} \cdot \mathrm{d}x.$$

On the sides of the tube the force f acts:

$$\mathrm{d}f = \pi r^2 (p_1 - p_2) = \pi r^2 \cdot \frac{\mathrm{d}p}{\mathrm{d}x} \cdot \mathrm{d}x.$$

If the flow is steady, the sum of these forces is 0 (remember that $\mathrm{d}p = \rho$):

$$2\eta l \cdot \frac{\mathrm{d}v}{\mathrm{d}r} = r \cdot \rho.$$

Since it's steady, $v = \text{const}, dv/dr = \text{const}, \Rightarrow \quad dp/dx = \text{const}.$

Thus,

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{\rho}{2\eta l} r,$$

Integrate it:

$$v = \frac{\rho}{4\eta l} r^2.$$

Per every 1 second volume dV passes through the pipe's cross-section of inner r and outer $r + dr$ radii:

$$\mathrm{d}V = 2\pi r \mathrm{d}r \cdot v,$$

$$V = \pi \frac{\rho}{2\eta l} \int_0^r r^2 \cdot r \cdot \mathrm{d}r = \pi \frac{\rho}{2\eta l} \cdot \frac{r^4}{4} = \frac{\pi}{8} \cdot \frac{\rho r^4}{\eta l}.$$

This is Poiseuille's law. However he got the expression by dimensional analysis.

Answer:

$$V = \frac{\pi}{8} \cdot \frac{\rho r^4}{\eta l}.$$