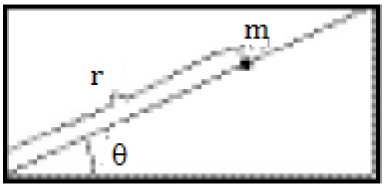
Answer on Question #74484, Physics / Mechanics | Relativity

A Particle of mass m rests on a plane with no friction. The plane is raised to an inclination angle θ with a constant rate α (θ(t=0) = 0), causing the particle to slide down the plane. Determine the motion of the particle.

Solution:



This problem is an example of a problem with a velocity-dependent constraint. However, if we can easily incorporate the constraint into the Lagrangian, we do not need to worry about constraint functions. In this example, we use our knowledge of the constraint immediately in our expression of the kinetic and the potential energy. Putting the origin of our coordinate system at the bottom of the plane we find

$$L = T + U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\sin\theta$$

where

$$\theta = \alpha t; \quad \dot{\theta} = \alpha$$

So,

$$L = \frac{1}{2}m(\dot{r}^2 + \alpha^2 r^2) - mgr\sin\alpha t$$

Lagrange's equation for *r* gives

$$m\ddot{r} = m\alpha^2 r - mg\sin\alpha t$$

or

$$\ddot{r} - \alpha^2 r = -g\sin\alpha t$$

The general solution is of the form $r = r_h + r_n$ where r_h is the general solution of the homogeneous $\ddot{r} - \alpha^2 r = -0$ and r_n is a particular solution. So

$$r_h = Ae^{\alpha t} + Be^{-\alpha t}$$

For r_n, try a solution of the form $r_p = C\sin\alpha t$. Then $\ddot{r}_p = -C\alpha^2\sin\alpha t$

Substituting into above equation gives

$$-C\alpha^2\sin\alpha t - -C\alpha^2\sin\alpha t = -g\sin\alpha t$$

$$C = \frac{g}{2\alpha^2}$$

So,

$$r(t) = Ae^{\alpha t} + Be^{-\alpha t} + \frac{g}{2\alpha^2}\sin\alpha t$$

We can determine A and B from the initial conditions:

$$r(0) = r_0$$
$$\dot{r}(0) = 0$$

So,

$$r_0 = A + B$$
$$0 = A - B + \frac{g}{2\alpha^2}$$

Solving for A and B gives:

$$A = \frac{1}{2}\left[r_0 - \frac{g}{2\alpha^2}\right] \qquad B = \frac{1}{2}\left[r_0 + \frac{g}{2\alpha^2}\right]$$

So, the analytic solution to this problem is

$$r(t) = \frac{1}{2}\left[r_0 - \frac{g}{2\alpha^2}\right] e^{\alpha t} + \frac{1}{2}\left[r_0 + \frac{g}{2\alpha^2}\right] e^{-\alpha t} + \frac{g}{2\alpha^2}\sin\alpha t$$

or

$$r(t) = r_0 \cosh(\alpha t) + \frac{g}{2\alpha^2}(\sin\alpha t - \sinh(\alpha t))$$

Answer: $r(t) = r_0 \cosh(\alpha t) + \frac{g}{2\alpha^2}(\sin\alpha t - \sinh(\alpha t))$.

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