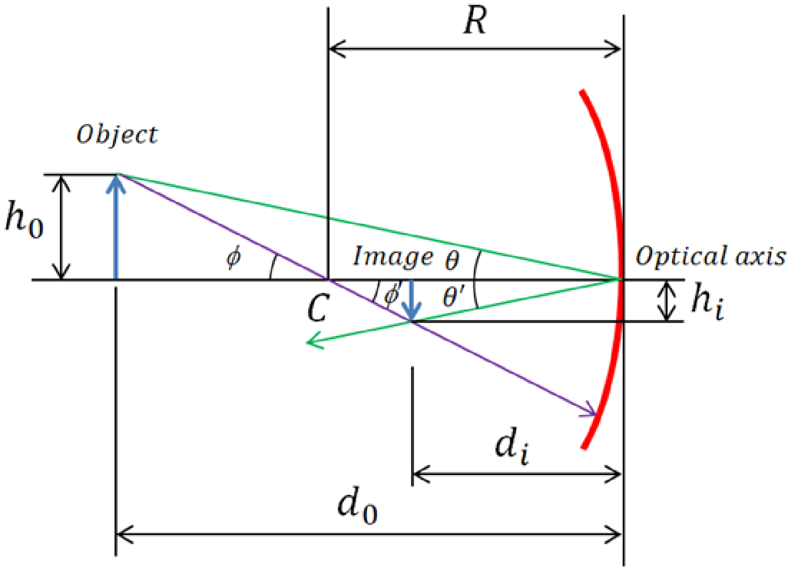
Answer on Question #72290, Physics / Optics

Question. Derive the mirror formula.What is the corresponding formula for a thin lens.

Solution.



Consider the object OP shown in Figure. The center of curvature of the mirror is labeled C and is a distance R from the vertex of the mirror, as marked in the figure. The object and image distances are labeled d_0 and d_i, and the object and image heights are labeled h_0 and h_i, respectively. Because the angles ϕ and ϕ' are alternate interior angles, we know that they have the same magnitude. However, they must differ in sign if we measure angles from the optical axis, so $\phi = -\phi'$. An analogous scenario holds for the angles θ and θ'. The law of reflection tells us that they have the same magnitude, but their signs must differ if we measure angles from the optical axis. Thus, $\theta = -\theta'$. Taking the tangent of the angles θ and θ', and using the property that $tg(-\theta) = -tg\,\theta$, gives us

$$\left.\begin{aligned} tg\,\theta &= \frac{h_0}{d_0} \\ tg\,\theta' = -tg\,\theta &= \frac{h_i}{d_i} \end{aligned}\right\} \frac{h_0}{d_0} = -\frac{h_i}{d_i} \quad or \quad -\frac{h_0}{h_i} = \frac{d_0}{d_i}$$

Similarly, taking the tangent of ϕ and ϕ' gives

$$\left.\begin{aligned} tg\,\phi &= \frac{h_0}{d_0 - R} \\ tg\,\phi' = -tg\,\phi &= \frac{h_i}{R - d_i} \end{aligned}\right\} \frac{h_0}{d_0 - R} = -\frac{h_i}{R - d_i} \quad or \quad -\frac{h_0}{h_i} = \frac{d_0 - R}{R - d_i}.$$

Combining these two results gives

$$\frac{d_0}{d_i} = \frac{d_0 - R}{R - d_i}$$

or

$$d_0(R - d_i) = d_i(d_0 - R) \;\rightarrow\; d_0R - d_0d_i = d_0d_i - d_iR \;\rightarrow\; 2d_0d_i = d_0R + d_iR \;\rightarrow$$

$$\frac{d_0 + d_i}{d_0d_i} = \frac{2}{R} \;\rightarrow\; \frac{1}{d_0} + \frac{1}{d_i} = \frac{2}{R} \quad (1).$$

No approximation is required for this result, so it is exact. However, in the small-angle approximation, the focal length of a spherical mirror is one-half the radius of curvature of the mirror, or $f = R/2$. Inserting this into Eq. 1 gives the *mirror equation or formula*:

$$\mathbf{\frac{1}{d_0} + \frac{1}{d_i} = \frac{1}{f}}.$$

The mirror equation relates the image and object distances to the focal distance and is valid only in the small-angle approximation. Although it was derived for a concave mirror, it also holds for convex mirrors. We can extend the mirror equation to the case of a plane mirror by noting that a plane mirror has an infinite radius of curvature. This means the focal point is at infinity, so the mirror equation simplifies to

$$\mathbf{d_0 = -d_i}.$$

As done for spherical mirrors, we can use ray tracing and geometry to show that, for a thin lens,

$$\frac{1}{d_0} + \frac{1}{d_i} = \frac{1}{f}.$$

where f is the focal length of the thin lens. This is the thin-lens equation.

(More detail see Max Born & Emil Wolf Principles of Optics)