Answer on Question #71428, Physics / Electromagnetism

A point charge q is brought to a position a distant d away from an infinite plane conductor held at zero potential. Use the method of images, to find:

i. The surface charge density induced on the plane;
ii. The force between the plane and the charge by using Coulomb's law for the force between the charge and its image;
iii. The total force acting on the plane by integrating.

Solution:

i.

$$\Phi_{+}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{x}-\vec{x}''|} = \frac{q}{4\pi\varepsilon_0\sqrt{(x^2+y^2+(z-d)^2}}$$

$$\Phi_{-}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{-q}{|\vec{x}-\vec{x}''|} = \frac{-q}{4\pi\varepsilon_0\sqrt{(x^2+y^2+(z+d)^2}}$$

$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0\left[\sqrt{(x^2+y^2+(z-d)^2}-\sqrt{(x^2+y^2+(z+d)^2}\right]}$$

$$\sigma = -\varepsilon_0\frac{\partial\Phi}{\partial z}\bigg|_{z=0} = \varepsilon_0\frac{q}{4\pi\varepsilon_0}\left[\frac{(z-d)}{((x^2+y^2+(z-d)^2)^{3/2}} - \frac{(z+d)}{((x^2+y^2+(z+d)^2)^{3/2}}\right]$$

$$\sigma = -\frac{qd}{2\pi}((x^2+y^2+d^2)^{-3/2} = -\frac{qd}{2\pi}((r^2+d^2)^{-3/2}$$

ii.

$$\vec{F} = q\vec{F} = q\frac{-q}{4\pi\varepsilon_0(2d)^2} = -\frac{q^2}{16\pi\varepsilon_0 d^2}\vec{z}$$

iii.

$$F = \int\frac{\sigma^2}{2\varepsilon_0}dA = \int\frac{\left(\frac{qd}{2\pi}\right)^2(r^2+d^2)^{-3}}{2\varepsilon_0}dA = \int_0^\infty\int_0^{2\pi} rdrd\varphi\frac{\left(\frac{qd}{2\pi}\right)^2(r^2+d^2)^{-3}}{2\varepsilon_0}$$

$$F = 2\pi\frac{\left(\frac{qd}{2\pi}\right)^2}{2\varepsilon_0}\int^\infty rdr(r^2+d^2)^{-3}$$

$$u = r^2+d^2$$

$$dr = \frac{du}{2r}$$

$$F = 2\pi\frac{\left(\frac{qd}{2\pi}\right)^2}{2\varepsilon_0}\int r\,u^{-3}\frac{du}{2r} = 2\pi\frac{\left(\frac{qd}{2\pi}\right)^2}{2\varepsilon_0}\frac{1}{2}\int du\,u^{-3} = 2\pi\frac{\left(\frac{qd}{2\pi}\right)^2}{2\varepsilon_0}\frac{1}{2}\left[-\frac{1}{2}(r^2+d^2)^{-2}\right]\Bigg|_0^\infty = \frac{q^2}{16\pi\varepsilon_0 d^2}$$