Answer on Question #70883 - Physics / Astronomy | Astrophysics

I learned Kepler's second law which was about equal areas and we PROVED that law of areas mathematically and also in his third law we PROVED, mathematically, that the period squared is equal to semi-major axis cubed, but in his first law which is:"All planets move about the Sun in elliptical orbits, having the Sun as one of the foci" we did not PROVE this law. My instructor just said that "as L is constant, which implies that the orbit lies in a plane!" It just says that the orbit lies in a plane, not that it is elliptical which is actually the law. Should not we be proving that the orbit is elliptical with Sun as on of the foci? Or am I missing something?

Solution:

According to the law of universal gravitation:

$$\boldsymbol{a} = \frac{d^2\boldsymbol{r}}{dt^2} = f(r)\hat{\boldsymbol{r}}$$

In a polar coordination system:

$$\frac{d\boldsymbol{r}}{dt} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}},$$

$$\frac{d^2\boldsymbol{r}}{dt^2} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$

In a coordination form:

$$\ddot{r} - r\dot{\boldsymbol{\theta}}^2 = f(r)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

Combining both equations:

$$r\frac{d\dot{\theta}}{dt} + 2\frac{dr}{dt}\dot{\theta} = 0$$

By simplifying:

$$\frac{d\dot{\theta}}{\dot{\theta}} = -2\frac{dr}{r}$$

After integration:

$$ln\dot{\theta} = -2\ln(r) + \ln(l)$$

$$\ln(l) = \ln(r^2) + ln\dot{\theta}$$

$$l = r^2\dot{\theta}$$

l is an angular moment l = **r**×**v**

Let:

$$r = \frac{1}{u}$$

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{d\theta}{dt}\frac{du}{d\theta} = -l\frac{du}{d\theta}$$

$$\ddot{r} = -l\frac{d}{dt}\frac{du}{d\theta} = -l\dot{\theta}\frac{d^2u}{d\theta^2} = -l^2u^2\frac{d^2u}{d\theta^2}$$

Thus, the equation of movement in $\hat{\boldsymbol{r}}$ direction is:

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{l^2u^2}f(\frac{1}{u})$$

According the universal gravitation law:

$$f\left(\frac{1}{u}\right) = f(r) = -\frac{GM}{r^2} = -GMu^2$$

As a result:

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{l^2}$$

The general solution of this equation is:

$$u = \frac{GM}{l^2}[1 + e \cdot \cos(\theta - \theta_0)]$$

By changing of u on 1/r and if $\theta_0 = 0$:

$$r = \frac{1}{u} = \frac{l^2/GM}{1 + e \cdot cos\theta}$$

Thus, we get an equation of a conical section with e in a focus. In this way the first Kepler law is originating from the second Newton law and the universal gravitation law.