Answer on Question #69717 – Physics – Classical Mechanics

Equations of motion for double spherical pendulum by Lagrangian mechanics.

Solution. Let us write the Lagrangian:

$$L = \frac{m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2)}{2} + \frac{m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2)}{2} - m_1 g z_1 - m_2 g z_2,$$

where

$$x_1 = l_1 \sin\theta_1 \cos\phi_1\,, \qquad x_2 = l_1 \sin\theta_1 \cos\phi_1 + l_2 \sin\theta_2 \cos\phi_2,$$

$$y_1 = l_1 \sin\theta_1 \sin\phi_1\,, \qquad y_2 = l_1 \sin\theta_1 \sin\phi_1 + l_2 \sin\theta_2 \sin\phi_2,$$

$$z_1 = l_1 \cos\theta_1\,, \qquad z_1 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

and

$$\dot{x}_1 = l_1 \cos\theta_1 \cos\phi_1\, \dot{\theta}_1 - l_1 \sin\theta_1 \sin\phi_1\, \dot{\phi}_1,$$

$$\dot{x}_2 = l_1 \cos\theta_1 \cos\phi_1\, \dot{\theta}_1 - l_1 \sin\theta_1 \sin\phi_1\, \dot{\phi}_1 + l_2 \cos\theta_2 \cos\phi_2\, \dot{\theta}_2 - l_2 \sin\theta_2 \sin\phi_2\, \dot{\phi}_2,$$

$$\dot{y}_1 = l_1 \cos\theta_1 \sin\phi_1\, \dot{\theta}_1 + l_1 \sin\theta_1 \cos\phi_1\, \dot{\phi}_1,$$

$$\dot{y}_2 = l_1 \cos\theta_1 \sin\phi_1\, \dot{\theta}_1 + l_1 \sin\theta_1 \cos\phi_1\, \dot{\phi}_1 + l_2 \cos\theta_2 \sin\phi_2\, \dot{\theta}_2 + l_2 \sin\theta_2 \cos\phi_2\, \dot{\phi}_2,$$

then we have for all $q = \phi_1, \phi_2, \theta_1, \theta_2$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

Answer:

$$\begin{aligned} &2l_1(2\cos[\theta_1]\theta_1{}'\phi_1{}' + \sin[\theta_1]\phi_1{}'') \\ &\quad + l_2(\sin[\theta_2]\sin[\phi_1 - \phi_2](\theta_2{}')^2 + 2\cos[\theta_2]\cos[\phi_1 - \phi_2]\theta_2{}'\phi_2{}' \\ &\quad + \sin[\theta_2]\sin[\phi_1 - \phi_2](\phi_2{}')^2 - \cos[\theta_2]\sin[\phi_1 - \phi_2]\theta_2{}'' \\ &\quad + \cos[\phi_1 - \phi_2]\sin[\theta_2]\phi_2{}'') = 0, \end{aligned}$$

$$\begin{aligned} &l_1(-\sin[\theta_1]\sin[\phi_1 - \phi_2](\theta_1{}')^2 + 2\cos[\theta_1]\cos[\phi_1 - \phi_2]\theta_1{}'\phi_1{}' - \sin[\theta_1]\sin[\phi_1 - \phi_2](\phi_1{}')^2 \\ &\quad + \cos[\theta_1]\sin[\phi_1 - \phi_2]\theta_1{}'' + \cos[\phi_1 - \phi_2]\sin[\theta_1]\phi_1{}'') \\ &\quad + l_2(2\cos[\theta_2]\theta_2{}'\phi_2{}' + \sin[\theta_2]\phi_2{}'') = 0, \end{aligned}$$

$$\begin{aligned} &l_1(\sin[2\theta_1](\theta_1{}')^2 + 2\sin[2\theta_1](\phi_1{}')^2 - (3 + \cos[2\theta_1])\theta_1{}'') \\ &\quad + 2\big(g\sin[\theta_1] \\ &\quad + \cos[\theta_1]l_2(\cos[\phi_1 - \phi_2]\sin[\theta_2](\theta_2{}')^2 - 2\cos[\theta_2]\sin[\phi_1 - \phi_2]\theta_2{}'\phi_2{}' \\ &\quad + \cos[\phi_1 - \phi_2]\sin[\theta_2](\phi_2{}')^2 - \cos[\theta_2]\cos[\phi_1 - \phi_2]\theta_2{}'' \\ &\quad - \sin[\theta_2]\sin[\phi_1 - \phi_2]\phi_2{}'')\big) = 0, \end{aligned}$$

$$\begin{aligned} &g\sin[\theta_2] + \frac{1}{2}l_2(\sin[2\theta_2](\phi_2{}')^2 - 2\theta_2{}'') \\ &\quad + \cos[\theta_2]l_1(\cos[\phi_1 - \phi_2]\sin[\theta_1](\theta_1{}')^2 + 2\cos[\theta_1]\sin[\phi_1 - \phi_2]\theta_1{}'\phi_1{}' \\ &\quad + \cos[\phi_1 - \phi_2]\sin[\theta_1](\phi_1{}')^2 - \cos[\theta_1]\cos[\phi_1 - \phi_2]\theta_1{}'' \\ &\quad + \sin[\theta_1]\sin[\phi_1 - \phi_2]\phi_1{}'') = 0. \end{aligned}$$