Question.

State and prove Noether Theorem.

Statement.

If the action of a given system is invariant under the infinitesimal transformation that changes q to $q + \delta q$, then, corresponding to this transformation there exist a law of conservation, and the conserved quantity, J, can be obtained only from the Lagrangian and the infinitesimal transformation.

Proof.

If the infinitesimal transformation $q' = q + \delta q$ is a symmetry of the action then [1]

$$\delta\mathcal{L}(q, \dot{q}, t) = \frac{d}{dt} f(q, t)$$

from the invariance of action and

$$\delta\mathcal{L}(q, \dot{q}, t) = \frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial\mathcal{L}}{\partial \dot{q}_i}\delta\dot{q}_i$$

because the transformation is infinitesimal.

Using the Euler-Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) - \frac{\partial\mathcal{L}}{\partial q_i} = 0$$

we can rewrite first two equations

$$\frac{d}{dt} f(q, t) = \frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial\mathcal{L}}{\partial \dot{q}_i}\delta\dot{q}_i = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)\delta q_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{d}{dt}(q_i) = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta q_i\right)$$

or

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta q_i - f(q, t)\right) = 0,$$

where $\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta q_i - f(q, t)$ is Noether's current J which is conserved quantity.

Sources.

[1] L.D. Landau, E.M. Lifshitz "Mechanics"