Answer on Question #62449 - Physics – Quantum Mechanics

The eigenvalues and eigenfunctions of a quantum mechanical operator A are denoted by a_n and ψ_n, respectively. If $f(x)$ denotes a function that can be expanded in the powers of x, show that:

$$f(A)\psi_n = f(a_n)\psi_n.$$

Solution:

ψ_n are eigenfunctions and a_n are eigenvalues of A:

$$A\psi_n = a_n\psi_n.$$

For some power of the operator A:

$$A^m\psi_n = A^{m-1}A\psi_n = A^{m-1}a_n\psi_n = a_nA^{m-2}A\psi_n = a_n^2A^{m-2}\psi_n = \cdots = a_n^{m-1}A\psi_n = a_n^m\psi_n.$$

Function $f(x)$ can be expanded in the powers of x, that is,

$$f(x) = \sum_{k=0}^{\infty}\frac{1}{k!}f^{(k)}(0)x^k.$$

We can define function of an operator as

$$f(A) = \sum_{k=0}^{\infty}\frac{1}{k!}f^{(k)}(0)A^k.$$

Then

$$f(A)\psi_n = \sum_{k=0}^{\infty}\frac{1}{k!}f^{(k)}(0)A^k\,\psi_n = \sum_{k=0}^{\infty}\frac{1}{k!}f^{(k)}(0)(A^k\psi_n) = \sum_{k=0}^{\infty}\frac{1}{k!}f^{(k)}(0)a_n^k\psi_n$$
$$= \left(\sum_{k=0}^{\infty}\frac{1}{k!}f^{(k)}(0)a_n^k\right)\psi_n = f(a_n)\psi_n.$$