Answer on Question #61826, Physics / Other

Establish the equation of motion of a damped oscillator. Solve it for a weakly damped oscillator and discuss the significance of the results.

Solution:

Newton's Law for a spring system with linear damping reads

$$-kx - bv = ma$$

for a block of mass *m* attached to a spring of constant *k* with damping coefficient *b*.

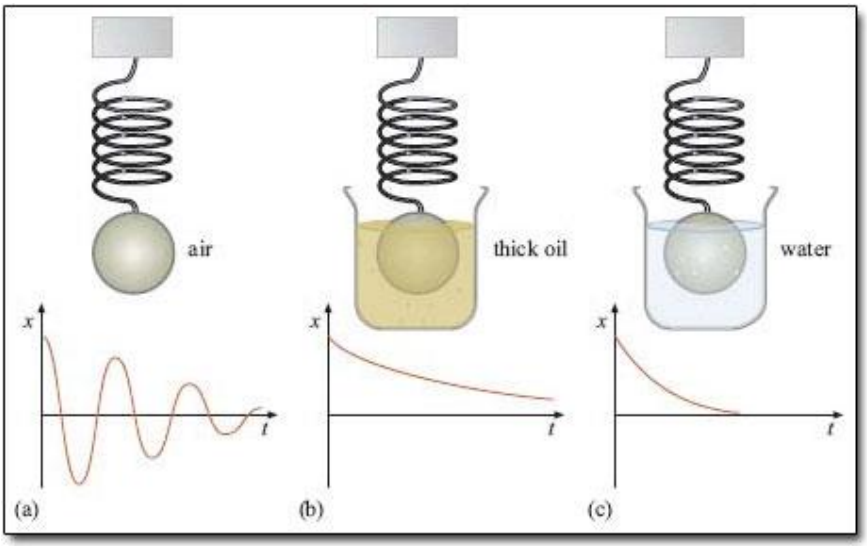

Figure: Plots of displacement vs time for the mass-spring system: (a) underdamped – mass in air; (b) overdamped – mass in thick oil; (c) critically damped – mass in water

Using the definitions of velocity and acceleration we can write this as the differential equation

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

This is just an homogeneous second order differential equation and has solution of the form

$$x(t) = Ae^{\lambda t}$$

Inserting this solution in (1) we can solve for λ . The following equation is obtained from the substitution:

$$m\lambda^2 + b\lambda + k = 0$$

Solving for [λ] we get

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

Now, it is clearly that our values for lambda depends on what's inside the square root. Since we wish to solve for a weakly damped oscillator, we have that

$$b^2 - 4mk < 0$$

This is also known as an under-damped oscillator. In this case lambda is a complex number. It's got a real part and an imaginary part.

The real part is -b/2m and we can figure out the imaginary part by writing $\sqrt{b^2 - 4mk}$ as

$$\sqrt{(-1)(4mk - b^2)} = i\sqrt{(4mk - b^2)}$$

So we can rewrite the solution as

$$\lambda = -\frac{b}{2m} \pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

The square root is the magnitude of the imaginary part. When b=0, the square root just becomes $\sqrt{k/m}$, the normal frequency of oscillation, so it makes sense to interpret this as a frequency

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

So

$$\lambda = -\frac{b}{2m} \pm i\omega$$

So are solution

$$x(t) = A_0 e^{-\frac{b}{2m}t} e^{\pm i\omega t}$$

Using Euler's equation [$e^{i\theta} = cos(\theta) + isin(\theta)$] and taking the real part we have:

$$x(t) = A_0 e^{-\frac{bt}{2m}} \cos(\omega_1 t + \varphi)$$

Looking at the solution for x(t), we can see that the amplitude A is modulated by the exponential, which is decreasing in time. So we have an oscillation whose amplitude is decreasing with time. That is, as t goes to infinity, our oscillation goes to zero because