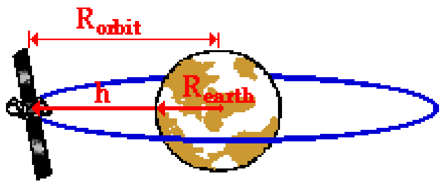
Answer on Question #61824, Physics / Other

i) A satellite revolves in a circular orbit around the earth at a certain height (h) above it. Suppose that h << R, the radius of the earth. Calculate the time period of revolution of the satellite.
ii) Derive expressions for average energy of a body executing SHM.

Solution:

(i)



This net centripetal force is the result of the gravitational force that attracts the satellite towards the central body and can be represented as

$$F_{grav} = \frac{(G * M_{sat} * M_{earth})}{R_{orbit}^2}$$

If the satellite moves in circular motion, then the net centripetal force acting upon this orbiting satellite is given by the relationship

$$F_{net} = \frac{(M_{sat} * v^2)}{R_{orbit}}$$

Since $F_{grav} = F_{net}$, the above expressions for centripetal force and gravitational force can be set equal to each other. Thus,

$$v^2 = \frac{(G * M_{earth})}{R_{orbit}}$$

Taking the square root of each side, leaves the following equation for the velocity of a satellite moving about a central body in circular motion

$$v = \sqrt{\frac{GM_{earth}}{R_{earth} + h}}$$

Period of revolution or time period of a satellite

It is the time taken by a satellite to complete one revolution around the Earth. Circumference of a circle of radius R + h is 2π(R+h).

Let T be the time period, v the velocity and h the height of the satellite above the Earth's surface.

$$T = \frac{2\pi(R + h)}{v}$$

$$v = \sqrt{\frac{GM}{R + h}}$$

Substituting in the equation, we get

$$T = \frac{2\pi(R+h)}{\sqrt{\frac{GM}{R+h}}}$$

$$T = \frac{2\pi(R+h)\sqrt{R+h}}{\sqrt{GM}} = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

The gravitational acceleration is

$$g = \frac{GM}{R^2}$$

Thus,

$$T = 2\pi\sqrt{\frac{(R+h)^3}{gR^2}}$$

$$T = \frac{2\pi}{R}\sqrt{\frac{(R+h)^3}{g}}$$

(ii)

Linear Harmonic oscillator.

Mechanical model: mass m on a spring characterized by a spring constant k.

Elastic restoring force $F = -kx$ is balanced according to Newton's second law

$$F = ma$$

$$m\ddot{x} = -kx$$

The equation of motion

$$\ddot{x} + \omega^2 x = 0$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

Such system oscillates with amplitude A and angular frequency ω.

Consider solution

$$x = A\sin(\omega t + \varphi)$$

The velocity v is equal to

$$v = \dot{x} = A\omega\cos(\omega t + \varphi)$$

and thus the kinetic energy

$$K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \varphi) = K_0\cos^2(\omega t + \varphi)$$

where the maximum kinetic energy is equal to

$$K_0 = \frac{1}{2}mA^2\omega^2 = \frac{1}{2}kA^2$$

The potential energy – work done by applied force displacing the system from 0 to x

$$U(x) = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

Substituting x

$$U(x) = \frac{1}{2}kA^2\sin^2(\omega t + \varphi) = U_0\sin^2(\omega t + \varphi)$$

where U_0 is the maximum potential energy (for x=A)

$$U_0 = \frac{1}{2}kA^2$$

The average values over one oscillation period are calculated using the definition

$$\langle f\rangle = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t)dt$$

Thus:

$$\langle U\rangle = \frac{\int_0^T Udt}{\int_0^T dt} = \frac{\int_0^T U_0\sin^2(\omega t + \varphi)\,dt}{T} = \frac{1}{2}U_0 = \frac{1}{2}kA^2$$

and

$$\langle K\rangle = \frac{\int_0^T Kdt}{\int_0^T dt} = \frac{1}{2}K_0 = \frac{1}{2}kA^2$$

The sum of the kinetic and potential energies in a simple harmonic oscillator is a constant, i.e., KE+PE=constant. The energy oscillates back and forth between kinetic and potential, going completely from one to the other as the system oscillates.

Thus,

$$\langle E\rangle = \langle K\rangle + \langle U\rangle = \frac{1}{2}kA^2 + \frac{1}{2}kA^2 = kA^2 = mA^2\omega^2$$