Answer on question 60941

The one-dimensional time-independent Schrödinger equation is

.

a/ A particle of mass m is contained in a one-dimensional box of width a. The potential energy U(x) is infinite at the walls of the box (x = 0 and x = a) and zero in between (0 < x < a).
Show that the solutions have the form: U(x)=Csin(n.pi.x/a) . Find the constant C.
b/ For the case n = 3, find the probability that the particle will be located in the region a/3 <x< 2a/3
c/ Sketch the wave-functions and the corresponding probability density distributions for the cases n = 1, 2 and 3.

Solution

$$\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + U(x)\right)\varphi(x) = E\varphi(x) \qquad (*)$$

a) Let check that the solutions have the form:

$$\varphi_n(x) = C_n \sin\left(\frac{n\pi x}{a}\right)$$

The particle can be located only between two wall (due to infinity of potential outside). So we have additional condition $\varphi_n(0) = \varphi_n(a) = 0$.
Using (*) we write:

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} C_n \sin\left(\frac{n\pi x}{a}\right) = \frac{\hbar^2}{2m}\left(\frac{\pi n}{a}\right)^2 \varphi_n(x) = E\varphi(x)$$

As we can from the last, the solutions have the needed form.
For finding C_n we use the condition:

$$\int_{-\infty}^{\infty} |\varphi_n(x)|^2 dx = 1 \to {C_n}^2 \int_{-a}^{a} \left|\sin\left(\frac{n\pi x}{a}\right)\right|^2 dx = \frac{{C_n}^2}{2}\int_{-a}^{a}\left(1-\cos\left(\frac{2n\pi x}{a}\right)\right)dx = a{C_n}^2$$

$$\to C_n = \boldsymbol{\frac{1}{\sqrt{a}}}$$

b) $P\left(x \in \left[\frac{a}{3};\frac{2a}{3}\right]\right) = \int_{\frac{a}{3}}^{\frac{2a}{3}} |\varphi_3(x)|^2 dx = \frac{1}{a}\int_{\frac{a}{3}}^{\frac{2a}{3}} \left|\sin\left(\frac{3\pi x}{a}\right)\right|^2 dx = \frac{1}{2a}\int_{\frac{a}{3}}^{\frac{2a}{3}}\left(1-\cos\left(\frac{6\pi x}{a}\right)\right)dx = \boldsymbol{\frac{1}{6}}$

c) (for scetching we take a=1)