Answer on Question #58205/ Physics – Molecular Physics | Thermodynamics

A rod 20 cm long, has both its ends maintained at 0*C at all times. At time t = 0 the temperature distribution in the rod is

T (x,0) = { 50*C for 0 < x < 10cm
{0*C for 10cm < x < 20cm

Obtain the temperature distribution u(x,t) in the rod if u(x,t) satisfies the diffusion equation:

d^2 u(x,t) / dx^2 = 0.5 (du(x,t)/dt)

Solution

Let us write down whole differential equation with boundary and initial conditions

u – is temperature of rod and function of x,t u=u(x,t), length of rod is l=20 cm

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}, 0 < x < l, t > 0$$

$$u(0,t) = 0$$
$$u(l,t) = 0$$

$$u(x,0) = \begin{cases} 50 = T_0, 0 < x < 10 \\ 0, 10 < x < 20 \end{cases}$$

So, we start from separation of variables.

Let $u(x,t) = X(x) \cdot T(t), where\ X(x) -\ function\ that\ depends\ only\ on\ x\ variable, T(t) - function\ that\ depends\ only\ on\ t\ variable.$

$$\frac{\partial^2 u}{\partial x^2} = T \cdot \frac{d^2 X}{dx^2}$$
$$\frac{\partial u}{\partial t} = X \cdot \frac{dT}{dt}$$

And

$$X(0) = 0$$
$$X(l) = 0$$

From initial equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}$$
$$T \cdot \frac{d^2 X}{dx^2} = \frac{1}{2} X \cdot \frac{dT}{dt}$$

Or

$$\frac{\frac{d^2 X}{dx^2}}{X} = \frac{1}{2}\frac{\frac{dT}{dt}}{T} = \lambda$$

And our task now is to find λ

$$\frac{\frac{d^2X}{dx^2}}{X} = \lambda$$

Here we see spectral problem with boundary conditions.

$$\frac{d^2X}{dx^2} = \lambda X$$

$$X(0) = 0$$
$$X(l) = 0$$

1) Put $\lambda = 0$

$$X = Cx + D$$
$$X(0) = 0 = D$$
$$X(l) = 0 = Cl \rightarrow C = 0$$

So, $\lambda = 0$ can't give us a solution

2) Put $\lambda > 0, \lambda = w^2$

$$\frac{d^2X}{dx^2} - w^2X = 0$$

Solution for this equation can be found in form of:

$$X = Csh(wx) + Dch(wx)$$
$$X(0) = 0 = C0 + D \rightarrow D = 0$$
$$X(l) = 0 = Csh(wl) \rightarrow or\ C = 0\ or\ sh(wl) = 0. If\ sh(wl) = 0 \rightarrow \lambda = 0, and\ initial\ assumption\ was\ \lambda > 0. If C = 0, again\ solution\ is\ trivial$$

3) Put $\lambda < 0, \lambda = -w^2$

$$\frac{d^2X}{dx^2} + w^2X = 0$$

Solution for this equation can be found in form of:

$$X = Csin(wx) + Dcos(wx)$$
$$X(0) = 0 = C0 + D \rightarrow D = 0$$
$$X(l) = 0 = Csin(wl), so\ if\ C \neq 0, then\ sin(wl) = 0$$

$$wl = n\pi, n = 1,2,3, \dots$$
$$w_n = \frac{n\pi}{l}$$

And here we have spectrum of eigenvalues.

Eigenfunctions are $X_n = C_n \sin(w_n x)$

$$\lambda = -w^2 = \frac{1}{2}\frac{\frac{dT}{dt}}{T}$$

From previous row: $T_n = D_n e^{-2w^2 t}$

So,

$$u_n(x,t) = \widetilde{D_n} e^{-2w_n^2 t} \sin(w_n x)$$

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(x,t)$$

Then, we put initial condition

$$u(x,0) = \sum_{n=1}^{+\infty} u_n(x,0) = \sum_{n=1}^{+\infty} \widetilde{D_n} \sin(w_n x)$$

And $\widetilde{D_n}$ are coefficients of Fourier transform

$$\widetilde{D_n} = \frac{2}{l}\int_0^l u(x,0)\sin(w_n x)dx$$

From

$$u(x,0) = \begin{cases} 50, 0 < x < 10 \\ 0, 10 < x < 20 \end{cases}$$

$$\widetilde{D_n} = \frac{2}{l}\int_0^{10} T_0 \sin(w_n x)\, dx = \frac{2T_0}{l}\left(-\frac{\cos(w_n x)}{w_n}\Big|_0^{10}\right) = \frac{2T_0 l}{l}\left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{n\pi}\Big|_{10=\frac{l}{2}}^{0}\right)$$

$$= \frac{2T_0}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right) = \frac{2T_0}{n\pi} 2sin^2\left(\frac{n\pi}{4}\right) = \frac{4T_0}{n\pi} sin^2\left(\frac{n\pi}{4}\right)$$

Result

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(x,t) = \sum_{n=1}^{+\infty} \frac{4T_0}{n\pi} sin^2\left(\frac{n\pi}{4}\right) e^{-2\left(\frac{n\pi}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right)$$