Answer on Question 57878, Physics, Mechanics, Relativity

Question:

A uniform rope of length L and mass M is held at one end and whirled in a horizontal circle with angular velocity ω. You can ignore the force of gravity on the rope. Find the time required for a transverse wave to travel from one end of the rope to the other.

Solution:

Let's consider a small segment of the rope between r and $r + \Delta r$. The segment has a length of Δr and a mass of $m = M\Delta r/L$. Let's treat this segment as a particle and consider the forces and the Newton's law on this segment. The inward force of tension from the rope to the segment is $T(r)$, and the outward force to the segment is $T(r + \Delta r)$. The acceleration of the segment is $r\omega^2$ and it points toward the pivot. Let's assume that the positive r direction is outward, then the acceleration is negative, and applying the Second Newton's law we get:

$$T(r + \Delta r) - T(r) = -mr\omega^2,$$

$$T(r + \Delta r) - T(r) = -\frac{M}{L}\Delta r r\omega^2,$$

$$\frac{T(r + \Delta r) - T(r)}{\Delta r} = -\frac{M}{L}r\omega^2.$$

Let Δr approach to zero, than, using the definition of derivative we get:

$$\frac{dT}{dr} = -\frac{M\omega^2}{L}r,$$

$$dT = -\frac{M\omega^2}{L}rdr,$$

After integration we get:

$$\int_{T_0}^{T(r)} dT = -\int_0^r \frac{M\omega^2}{L}rdr,$$

$$T = -\frac{M\omega^2}{L}\frac{r^2}{2} + C,$$

here, C is an integration constant and can be determined by considering the constraint that the force of tension T is zero at the tip of the rope, so $T = 0$ at $r = L$ and we get:

$$C = \frac{M\omega^2}{2}L,$$

$$T = -\frac{M\omega^2}{L}\frac{r^2}{2} + \frac{M\omega^2}{2}L = \frac{M\omega^2}{2L}(L^2 - r^2),$$

$$T = \frac{M\omega^2}{2L}(L^2 - r^2).$$

Then, we can find the velocity of the transverse wave in the rope from the formula:

$$v = \sqrt{\frac{T}{\mu}},$$

here, T is the force of tension in the rope, $\mu = M/L$ is the linear mass density of the rope.

Substituting μ into the previous formula we can find the velocity of the transverse wave in the rope:

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{\frac{M\omega^2}{2L}(L^2 - r^2)}{\frac{M}{L}}} = \sqrt{\frac{\omega^2}{2}(L^2 - r^2)},$$

$$\frac{dr}{dt} = \sqrt{\frac{\omega^2}{2}(L^2 - r^2)} = \frac{\omega}{\sqrt{2}}\sqrt{(L^2 - r^2)},$$

$$\frac{dr}{\sqrt{(L^2 - r^2)}} = \frac{\omega}{\sqrt{2}}dt,$$

$$\int_0^L \frac{dr}{\sqrt{(L^2 - r^2)}} = \frac{\omega}{\sqrt{2}}\int_0^t dt,$$

$$arcsin1 = \frac{\omega}{\sqrt{2}}t,$$

$$\frac{\pi}{2} = \frac{\omega}{\sqrt{2}} t,$$

$$t = \frac{\pi}{\sqrt{2} \cdot \omega}.$$

Answer:

$t = \frac{\pi}{\sqrt{2} \cdot \omega}$