Answer on Question55157 - Physics / Mechanics — Kinematics — Dynamics - for correction

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Find, by the Fourier-series method, the steady-state solution for an undamped harmonic oscillator subject to a force having the form of a rectified sin-wave: $F(t) = F|sin(\omega_0 t)|$, where is the natural frequency of the oscillator.

Solution

We want to obtain the steady-state solution to

$$mx'' + kx = F_0|sin(\omega_0 t)|$$

where ω_0 is the natural frequency of the oscillator.
Let's obtain the complex Fourier series representation for the driver

$$F_0|sin(\omega_0 t)| = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega t}$$

The period of $F_0|sin(\omega_0 t)|$ is $\frac{\pi}{\omega_0}$. The frequency is $2\pi/T = 2\omega_0$
Let's find the coefficients of the series.

$$c_n = \frac{\omega_0}{Pi}\int_0^{\pi/\omega_0} F_0 sin(\omega_0 t)e^{-2in\omega_0 t}dt =$$

$$= \frac{\omega_0 F_0}{2\pi i}\int_0^{\pi/\omega_0}(e^{i\omega_0 t} - e^{-i\omega_0 t})e^{-2in\omega_0 t} =$$

$$= \frac{\omega_0 F_0}{2\pi i}\int_0^{\pi/\omega_0}(\frac{e^{i\omega_0 t(1-2n)}}{i\omega_0(1-2n)} + \frac{e^{-i\omega_0 t(2n+1)}}{i\omega_0(2n+1)}) =$$

$$= \frac{2F_0}{\pi(1-4n^2)}$$

The solution of the problem $x(t)$ can be represented as the following sum:

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n(t)$$

where $x_n(t)$ is the solution of this equation:

$$mx_n'' + kx_n = c_n e^{in\omega t}$$

General solution of this problem looks like $x_n = a_n e^{in\omega t}$. After substitution $x_n = a_n e^{in\omega t}$ into the left side we find:

$$a_n = \frac{c_n}{k(1-2n^2)} = \frac{2F_0}{\pi k(1-4n^2)(1-2n^2)}$$

The final solution of the initial equation is:

$$x(t) = \sum_{n=-\infty}^{+\infty} \frac{c_n}{k(1-2n^2)} \frac{2F_0}{\pi k(1-4n^2)(1-2n^2)} e^{in\omega t}$$