Answer on Question #51444, Physics Solid State Physics

Consider a quantum particle confined in a well of width a. If the particle is in its ground state calculate the quantity ΔxΔp where $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$.

Solution:

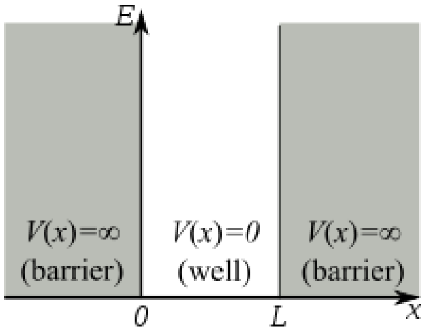

Fig.1

The wave function of a one-dimensional potential well

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right)$$

where x is the threading coordinate; L is length of the box; n is the level number.

The momentum operator

$$\hat{P}_X = -i\hbar \frac{\partial}{\partial x}$$

The average value of P_X if the particle is in its ground state

$$\left\langle P_X\right\rangle=\int_0^L \psi_1^*(x)\hat{P}_X\psi_1(x)\,dx=\int_0^L\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\left(-i\text{h}\frac{\partial}{\partial x}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)dx=-\frac{2i\text{h}}{L}\int_0^L\sin\left(\frac{\pi x}{L}\right)\left(\frac{\partial}{\partial x}\right)\sin\left(\frac{\pi x}{L}\right)dx=$$

$$-\frac{2i\text{h}}{L}\frac{\pi}{L}\int_0^L\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)dx=-\frac{i\text{h}\pi n}{L^2}\int_0^L\sin\left(\frac{2\pi n x}{L}\right)dx=-\frac{i\text{h}\pi n}{L^2}\cdot\frac{L}{2\pi n}\cos\left(\frac{2\pi n x}{L}\right)\Big|_0^L=0$$

The average value of $P_X{}^2$ if the particle is in its ground state

$$\left\langle P_X{}^2\right\rangle=\int_0^L \psi_1^*(x)\hat{P}_X{}^2\psi_1(x)\,dx=\int_0^L\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\left(-i\text{h}\frac{\partial}{\partial x}\right)^2\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)dx=$$

$$-\frac{2\text{h}^2}{L}\int_0^L\sin\left(\frac{\pi x}{L}\right)\left(\frac{\partial^2}{\partial x^2}\right)\sin\left(\frac{\pi x}{L}\right)dx=\frac{2\text{h}^2}{L}\left(\frac{\pi}{L}\right)^2\int_0^L\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)dx=\frac{\text{h}^2\pi^2}{L^2}$$

The coordinate operator

$$\hat{x}=x$$

The average value of $\hat{x}$ if the particle is in its ground state

$$\left\langle x\right\rangle=\int_0^L \psi_1^*(x)\hat{x}\psi_1(x)\,dx=\int_0^L\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)x\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)dx=L/2$$

The average value of x^2 if the particle is in its ground state

$$\left\langle x^2\right\rangle=\int_0^L \psi_1^*(x)x^2\psi_1(x)\,dx=\int_0^L\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)x^2\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)dx=\frac{L^2}{6}\left(2-3/\pi^2\right)$$

Then

$$\Delta x\Delta p=\sqrt{\left(\frac{\text{h}^2\pi^2}{L^2}-0\right)\left(\frac{L^2}{6}\left(2-3/\pi^2\right)-\frac{L^2}{4}\right)}=\frac{\text{h}}{2\sqrt{3}}\sqrt{\pi^2-6}$$

Answer: $\Delta x\Delta p=\frac{\text{h}}{2\sqrt{3}}\sqrt{\pi^2-6}$