Answer on Question #51443, Physics, Solid State Physics

Obtain the expectation value of the potential energy $V(x)=\frac{m\omega^2x^2}{2}$ of the one dimensional harmonic oscillator in the first excited state $\psi_1(x)=\frac{a^{3/2}\sqrt{2}}{\pi^{1/4}}x\exp\left[-\frac{a^2x^2}{2}\right]$

Solution:

The expectation value of the potential energy $V(x)=\frac{m\omega^2x^2}{2}$ of the one dimensional harmonic oscillator in the first excited state is given by Eq.(1)

$$\langle\psi_1|V|\psi_1\rangle=\int_{-\infty}^{+\infty}\psi_1(x)V(x)\psi_1(x)dx \tag{1}$$

So,

$$\langle\psi_1|V|\psi_1\rangle=\int_{-\infty}^{+\infty}\frac{a^{3/2}\sqrt{2}}{\pi^{1/4}}x\exp\left[-\frac{a^2x^2}{2}\right]\frac{m\omega^2x^2}{2}\frac{a^{3/2}\sqrt{2}}{\pi^{1/4}}x\exp\left[-\frac{a^2x^2}{2}\right]dx=\frac{2a^3}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{m\omega^2x^4}{2}\exp\left[-a^2x^2\right]dx=$$

$$\frac{m\omega^2}{a^2\sqrt{\pi}}\int_{-\infty}^{+\infty}(ax)^4\exp\left[-a^2x^2\right]d(ax)=\left|\begin{matrix}y=(ax)^2\\ dy=2axd(ax)=2\sqrt{y}d(ax)\end{matrix}\right|=\frac{m\omega^2}{a^2\sqrt{\pi}}\int_{-\infty}^{+\infty}y^2\exp[-y]\frac{dy}{2\sqrt{y}}=$$

$$=\frac{m\omega^2}{a^2\sqrt{\pi}}\int_{-\infty}^{+\infty}y^2\exp[-y]\frac{dy}{2\sqrt{y}}=\frac{m\omega^2}{2a^2\sqrt{\pi}}\int_{-\infty}^{+\infty}y^{3/2}\exp[-y]dy=\frac{m\omega^2}{2a^2\sqrt{\pi}}\Gamma(5/2)=$$

$$\frac{m\omega^2}{2a^2\sqrt{\pi}}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)=\frac{3m\omega^2}{8a^2\sqrt{\pi}}\cdot\sqrt{\pi}=\frac{3m\omega^2}{8a^2}$$

where $\Gamma(\xi)$ is the Gamma-function.

So,

$$\langle\psi_1|V|\psi_1\rangle=\frac{3m\omega^2}{8a^2} \tag{2}$$

Answer: $\langle\psi_1|V|\psi_1\rangle=\frac{3m\omega^2}{8a^2}$