Answer on Question #51341, Physics, Electric Circuits

Determine the resonant frequency for the circuit :
1)10 ohm
2)AC input
3)1 mH
4)100 nF
Calculate the impedance of the circuit 5 kHz above and 5 kHz below the resonance frequency.

Answer:

Assume that we have a deal with RLC linear circuit
http://en.wikipedia.org/wiki/RLC_circuit

In this circuit, the three components are all in series with the voltage source. The governing differential equation can be found by substituting into Kirchhoff's voltage law (KVL) the constitutive equation for each of the three elements. From KVL,

$$v_R + v_L + v_C = v(t)$$

where v_R, v_L, v_C are the voltages across R, L and C respectively and $v(t)$ is the time varying voltage from the source. Substituting in the constitutive equations,

$$Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{\tau=t} i(\tau)\, d\tau = v(t)$$

For the case where the source is an unchanging voltage, differentiating and dividing by L leads to the second order differential equation:

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0$$

This can usefully be expressed in a more generally applicable form:

$$\frac{d^2i(t)}{dt^2} + 2\alpha\frac{di(t)}{dt} + {\omega_0}^2 i(t) = 0$$

α and ω_0 are both in units of angular frequency. α is called the *neper frequency*, or *attenuation*, and is a measure of how fast the transient response of the circuit will die away after the stimulus has been removed. Neper occurs in the name because the units can also be considered to be nepers per second, neper being a unit of attenuation. ω_0 is the angular resonance frequency.[3]

For the case of the series RLC circuit these two parameters are given by:[4]

$$\alpha = \frac{R}{2L} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}$$

A useful parameter is the *damping factor*, ζ which is defined as the ratio of these two,

$$\zeta = \frac{\alpha}{\omega_0}$$

In the case of the series RLC circuit, the damping factor is given by,

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

The value of the damping factor determines the type of transient that the circuit will exhibit.[5] Some authors do not use ζ and call α the damping factor.[6]

The differential equation for the circuit solves in three different ways depending on the value of ζ. These are underdamped ($\zeta<1$), overdamped ($\zeta>1$) and critically damped ($\zeta=1$). The differential equation has the characteristic equation

$$s^2 + 2\alpha s + {\omega_0}^2 = 0$$

The roots of the equation in *s* are

$$s_1 = -\alpha + \sqrt{\alpha^2 - {\omega_0}^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - {\omega_0}^2}$$

The general solution of the differential equation is an exponential in either root or a linear superposition of both,

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

The coefficients A_1 and A_2 are determined by the boundary conditions of the specific problem being analysed. That is, they are set by the values of the currents and voltages in the circuit at the onset of the transient and the presumed value they will settle to after infinite time.[8]

Underdamped response

The underdamped response ($\zeta<1$) is

$$i(t) = B_1 e^{-\alpha t}\cos(\omega_d t) + B_2 e^{-\alpha t}\sin(\omega_d t)$$

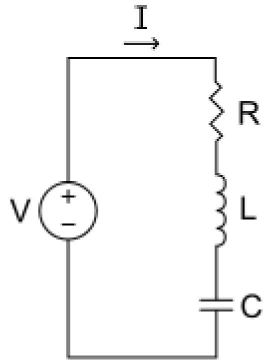
By applying standard trigonometric identities the two trigonometric functions may be expressed as a single sinusoid with phase shift,[12]

$$i(t) = B_3 e^{-\alpha t}\sin(\omega_d t + \varphi)$$

The under damped response is a decaying oscillation at frequency ω_d. The oscillation decays at a rate determined by the attenuation α. The exponential in α describes the envelope of the oscillation. B_1 and B_2 (or B_3 and the phase shift φ in the second form) are arbitrary constants determined by boundary conditions. The frequency is given by

$$\omega_d = \sqrt{{\omega_0}^2 - \alpha^2} = \omega_0\sqrt{1-\zeta^2}$$

This is called the damped resonance frequency or the damped natural frequency. It is the frequency the circuit will naturally oscillate at if not driven by an external source. The resonance frequency, ω_0, which is the frequency at which the circuit will resonate when driven by an external oscillation, may often be referred to as the undamped resonance frequency to distinguish it.



The resonant frequency defined for this circuit as:

$$= \frac{w}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{1\,mH \cdot 100\,nF} - \frac{(10\,Om)^2}{4(1\,mH)^2}} \approx 16\,kHz$$

The resistance is:

$$Z(w) = R + i(wL - \frac{1}{wC})$$

$Z(w_r + 5kHz) = Z(21\,kHz) = 10 + i\left(2\pi \cdot 21 \cdot 10^3\,Hz\ \cdot 10^{-3}\,H - \frac{1}{2\pi \cdot 21 \cdot 10^3\,Hz 10^{-7} F}\right) = 10 + 56.16\,i\,Om$

$$Z(w_r - 5kHz) = Z(11\,kHz) = 10 + i\left(2\pi \cdot 11 \cdot 10^3\,Hz\ \cdot 10^{-3}\,H - \frac{1}{2\pi \cdot 11 \cdot 10^3\,Hz 10^{-7} F}\right) = 10 - 75.57\,i\,Om$$