Answer on Question #51224, Physics, Other

Consider a quantum particle confined in a well of width a. If the particle is in its ground state calculate the quantity ΔxΔp where $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$.

Solution:

The wave function of a one-dimensional potential well is given by Eq.(1)

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right) \quad (1)$$

where L is length of the box.

The momentum operator

$$\hat{P}_X = -i\hbar \frac{\partial}{\partial x} \quad (2)$$

Find the average value of P_X

$$\langle P_X \rangle = \int_0^L \psi_n^*(x) \hat{P}_X \psi_n(x)\, dx = \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right)\left(-i\hbar \frac{\partial}{\partial x}\right)\sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right) dx =$$
$$-\frac{2i\hbar}{L}\int_0^L \sin\left(\frac{\pi n x}{L}\right)\left(\frac{\partial}{\partial x}\right)\sin\left(\frac{\pi n x}{L}\right) dx = -\frac{2i\hbar}{L}\frac{\pi n}{L}\int_0^L \sin\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi n x}{L}\right) dx =$$
$$-\frac{i\hbar\pi n}{L^2}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) dx = -\frac{i\hbar\pi n}{L^2}\cdot\frac{L}{2\pi n}\cos\left(\frac{2\pi n x}{L}\right)\Bigg|_0^L = -\frac{i\hbar}{2L}\left(\cos\left(\frac{2\pi n L}{L}\right) - cos0\right) =$$
$$= -\frac{i\hbar}{2L}(1-1) = 0$$

Find the average value of $P_X{}^2$

$$\langle P_X{}^2 \rangle = \int_0^L \psi_n^*(x) \hat{P}_X{}^2 \psi_n(x)\, dx = \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right)\left(-i\hbar \frac{\partial}{\partial x}\right)^2 \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right) dx =$$
$$-\frac{2\hbar^2}{L}\int_0^L \sin\left(\frac{\pi n x}{L}\right)\left(\frac{\partial^2}{\partial x^2}\right)\sin\left(\frac{\pi n x}{L}\right) dx = \frac{2\hbar^2}{L}\left(\frac{\pi n}{L}\right)^2\int_0^L \sin\left(\frac{\pi n x}{L}\right)\sin\left(\frac{\pi n x}{L}\right) dx =$$
$$= \frac{\hbar^2 n^2 \pi^2}{2L^2}\left(2 - \frac{\sin[2n\pi]}{n\pi}\right) = \frac{\hbar^2 n^2 \pi^2}{L^2}$$

The coordinate operator

$$\hat{x} = x \quad (3)$$

Find the average value of $\hat{x}$ (n=1)

$$\langle x\rangle = \int_0^L \psi_n^*(x)\hat{x}\psi_n(x)\,dx = \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)x\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)dx =$$

$$\frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)x\sin\left(\frac{\pi x}{L}\right)dx = L/2$$

Find the average value of x^2 (n=1)

$$\langle x^2\rangle = \int_0^L \psi_n^*(x)x^2\psi_n(x)\,dx = \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)x^2\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)dx =$$

$$\frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)x^2\sin\left(\frac{\pi x}{L}\right)dx = \frac{L^2}{6}\left(2-3/\pi^2\right)$$

Than

$$\Delta x\Delta p = \sqrt{\left(\frac{\hbar^2\pi^2}{L^2}-0\right)\left(\frac{L^2}{6}\left(2-3/\pi^2\right)-\frac{L^2}{4}\right)} = \frac{\hbar}{2\sqrt{3}}\sqrt{\pi^2-6}$$

Answer:

$$\Delta x\Delta p = \frac{\hbar}{2\sqrt{3}}\sqrt{\pi^2-6}$$