Answer on Question #51223-Physics-Other

Obtain the expectation value of the potential energy $V\,(x) = \frac{m\omega^2 x^2}{2}$ of the one dimensional harmonic oscillator in the first excited state

$$\psi_1(x) = 2\sqrt{\frac{a}{2\sqrt{\pi}}}axe^{-\frac{a^2x^2}{2}}$$

Solution

$$\langle\psi_1|V|\psi_1\rangle = \int_{-\infty}^{\infty}\psi_1(x)V(x)\psi_1(x)dx = \int_{-\infty}^{\infty}\left(2\sqrt{\frac{a}{2\sqrt{\pi}}}axe^{-\frac{a^2x^2}{2}}\right)^2\frac{m\omega^2x^2}{2}dx$$

$$= \int_{-\infty}^{\infty}4\frac{a}{2\sqrt{\pi}}(ax)^2\frac{m\omega^2x^2}{2}e^{-a^2x^2}dx = \int_{-\infty}^{\infty}\frac{a}{\sqrt{\pi}}(ax)^2m\omega^2x^2e^{-a^2x^2}dx = |ax = y|$$

$$= \int_{-\infty}^{\infty}\frac{m}{\sqrt{\pi}}\left(\frac{\omega}{a}\right)^2y^4e^{-y^2}dy = \frac{m}{\sqrt{\pi}}\left(\frac{\omega}{a}\right)^2\int_{-\infty}^{\infty}y^4e^{-y^2}dy.$$

$$\int_{-\infty}^{\infty}y^4e^{-y^2}dy = |y^2 = z| = \int_{-\infty}^{\infty}z^2e^{-z}d\sqrt{z} = \frac{1}{2}\int_{-\infty}^{\infty}z^{\frac{3}{2}}e^{-z}dz = \frac{1}{2}\Gamma\left(\frac{5}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{8}\sqrt{\pi}.$$

where $\Gamma(x)$ is the gamma-function.

Thus,

$$\langle\psi_1|V|\psi_1\rangle = \frac{m}{\sqrt{\pi}}\left(\frac{\omega}{a}\right)^2\frac{3}{8}\sqrt{\pi} = \frac{3}{8}\frac{m\omega^2}{a^2}.$$

Answer: $\frac{3}{8}\frac{m\omega^2}{a^2}$.