**Answer on Question #51192, Physics, Mechanics, Kinematics, Dynamics**

What is the Mathematical proof of free and forced oscillations?

Solution:

If any physical quantity that characterizes the movement accepts duplicate values, such a movement is called oscillation. Fluctuations can be periodic or non-periodic. If the oscillation is periodic, then the dynamic variable takes duplicate values at regular intervals, which are called period.

Periodic oscillations can be harmonic or not harmonic. Harmonic oscillations such vibrations in which the dynamic quantity varies harmonically, i.e. it changes over time and characterized by a sinusoidal or cosine law. All others oscillations are called non-harmonic.

Now we can derive an equation which expresses the motion of a material point performs harmonic oscillations. The simplest oscillating system is a harmonic oscillator. Dimensional harmonic oscillator model is the system shown in Figure 1

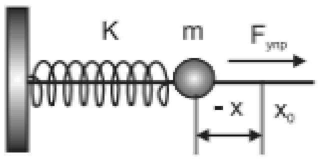



Figure 1 Model-dimensional harmonic oscillator.

The spring stiffness K is attached to one end of the vertical wall. On the other point placed the ball of mass m, which can slide without friction on a smooth horizontal rod. If the elastic force of the spring obeys Hooke's law $F_\tau = -Kx$, is shown on Figure 1, the system is precisely the model of one-dimensional harmonic oscillator.

In this example, the mass m moves solely under the influence of quasi-elastic force $F_\tau$, since the force of gravity is compensated by the reaction force bearing, and there is no friction. We write the equation of Newton dynamics for the mass m: $ma = -Kx$. Here is the acceleration of a material point, and x - its movement. The origin of the coordinate system is chosen at the point $x_0$, which corresponds to the not deformed state of the spring. Tensile spring x-coordinate is positive in compression - is negative.

Since we have

$$a = \frac{\Delta v}{\Delta t} = \frac{\Delta}{\Delta t}\frac{\Delta x}{\Delta t} = \frac{\Delta^2 x}{(\Delta t)^2}$$

An acceleration is the second derivative of the displacement x and time t, the equation of motion of a harmonic oscillator can be written in the form.

$$m\frac{d^2x}{dt^2} = -Kx \text{ or } m\frac{d^2x}{dt^2} + Kx = 0$$

Now we introduce a more compact notation for the first and second derivatives, assuming $v = x'$, $a = x''$. Then the equation of motion for a harmonic oscillator has the following form.

$$mx'' + Kx = 0$$

In mathematical classification the noted above equation describes the natural oscillations of the oscillator is linear homogeneous second order differential equation with constant coefficients.

Equation is linear and homogeneous, since all its terms contain the unknown quantity x in the first degree. The resulting equation of the second order with constant coefficients, as the leading derivative with respect to time is of second order, and the coefficients of the unknowns are independent of time. We obtain the solution of this equation is a simple selection of a suitable functional dependence for x (t). We seek a solution of the trial as follows:

$$x(t) = C_1 \cdot \sin(wt)$$

Where $C_1$ and $w$ are unknown constants.

We substitute the trial solution into the original equation and using the rules for calculating the second derivative of the function $\sin(wt)$. We obtain the following result.

$$-w^2 m + K = 0$$

The resulting equation is called the characteristic and allows us to find the natural frequency of vibration of the system.

$$w_0 = \sqrt{\frac{K}{m}}$$

This mean the expression will be equal to

$$x(t) = C_2 \cdot \cos(w_0 t)$$

It also is a particular solution of the original equation for arbitrary values of the constant $C_2$. Therefore, it can be argued that the most common form of the solution of the equation is the following

$$x(t) = C_1 \sin(w_0 t) + C_2 \cdot \cos(w_0 t)$$

Thus, the general solution of the problem of oscillations of a one-dimensional harmonic oscillator contains two arbitrary constants $C_1$ and $C_2$, which can be found from the initial conditions. We can override the constants, write the solution in a more convenient form. Indeed, by introducing new constants $x_0$ and δ; $C_1 = x_0 \cos(\delta)$, $C_2 = x_0 \sin(\delta)$,, we can write the general solution of equation in the following form.

$$x(t) = x_0 \sin(w_0 t + \delta)$$

Where $x_0$the oscillation amplitude and δ the initial phase of the oscillation.

The oscillation amplitude characterizes the maximum deviation of the oscillating system from the equilibrium position. Phase fluctuations are the quantity $\omega_0 t + \delta$, which characterizes the state of the oscillation process to an arbitrary time t. We find the relation of the oscillation period T with frequency $\nu = \frac{w_0}{2\pi}$. Since the period T - is the time during which one complete oscillation takes place, that must be satisfied $w_0 T = 2\pi$. Hence, for the period of oscillation of the harmonic oscillator we obtain

$$T = \frac{1}{\nu} = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{K}}$$

Constants $x_0$ and δ are determined from the initial conditions of excitation of oscillations, which may changes.

Let all transients have been completed and the system remained only oscillations with a frequency of the driving force. We seek a solution in the form.

$$x(t) = x_0 \cos(wt - \delta)$$

Where, $x_0$ the unknown oscillation amplitude, and δ - unknown initial phase. We calculate the value of speed and acceleration to move is given by

$$x'(t) = -x_0 w \cdot \sin(wt - \delta) = x_0 w \cdot \cos(wt - \delta + \frac{\pi}{2})$$

$$x''(t) = -x_0 w^2 \cos(wt - \delta)$$

If we have deal with damping and driving force, then the equation of motion of a harmonic oscillator has the form.

$$x'' + 2\gamma x' + w_0 x^2 = \frac{f_0}{m}\cos(wt)$$

This equation, unlike the equations for own oscillations and damped oscillations, is not uniform, since the right side of the equation does not depend on the variable x. The general solution of this equation is the sum of the general solution of the homogeneous equation

$$x'' + 2\gamma x' + w_0 x^2 = 0$$

As we have already established, the natural oscillations in the presence of the force of friction damped in a fairly short time on the order of time τ, and so after a while there will be only oscillations with a frequency of the driving force ω. Thus, the forced oscillations do not occur at the natural frequency of oscillation $w_0$, but at the frequency of the driving force.

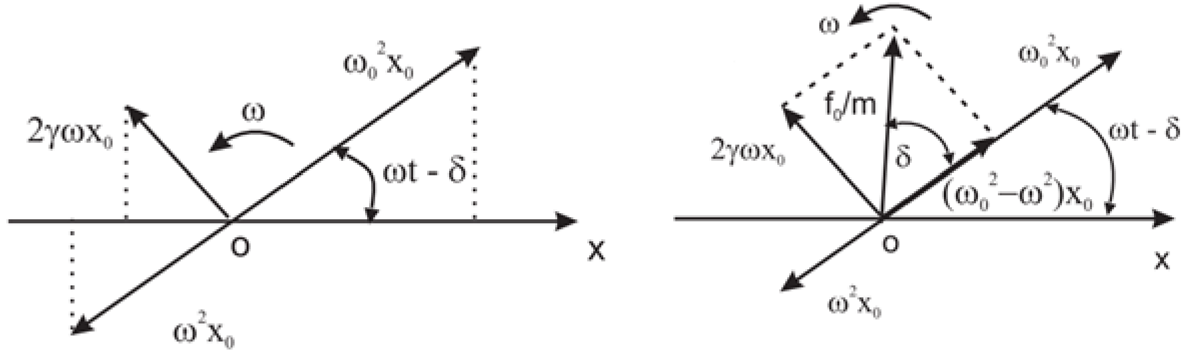
Now we back to problem noted above. We substitute $x'(t) = -x_0 w \cdot \sin(wt - \delta) = x_0 w \cdot \cos(wt - \delta + \frac{\pi}{2})$ and equation $x''(t) = -x_0 w^2 \cos(wt - \delta)$ into equation $x'' + 2\gamma x' + w_0 x^2 = \frac{f_0}{m}\cos(wt)$. Thus we obtain the equation for determining the amplitude and phase of forced oscillations.

$$-w^2x_0\cos(wt-\delta)+2\gamma wx_0\cos\left(wt-\delta+\frac{\pi}{2}\right)+w_0^2x_0\cos(wt-\delta)=\frac{f_0}{m}\cos(wt)$$

We note an important feature of the obtained equation: all terms of this equation changes with time according to the harmonic law, and the phase shift remains constant in time. To determine the amplitude and phase in the forced oscillation case, we apply the method of vector diagrams.

Taking into account that the instantaneous value of any quantity varying harmonic, the projection can be represented by the corresponding vector rotating with a constant angular velocity ω, and represent each of the terms on the left-hand side of the equation in the form of a rotating vector. Projection of the vector $w_0^2x_0$ matches to the $w_0^2x_0\cos(wt-\delta)$, the vector of $w^2x_0$ is phase-shifted by π and corresponds to the term of $-w^2x_0\cos(wt-\delta)$. Vector of $2\gamma wx_0$ ahead of the vector $w_0^2x_0$ phase by $\frac{\pi}{2}$ corresponds in the equation $2\gamma wx_0\cos\left(wt-\delta+\right.$ $\left.+\frac{\pi}{2}\right)$.

The entire set of vectors is rotated with an angular velocity ω, as shown in Figure 2a. Since the sum of the projections of the vectors is equal to the total projection of the vector, it is possible for all the vector of the diagram on the Figure 2a lay down according to the rules of vector addition. Then, instead of the three vectors Figure 2a, we obtain a vector (Figure 2b), which is equal to the right side of the equation $-w^2x_0\cos(wt-\delta)+2\gamma wx_0\cos\left(wt-\delta+\frac{\pi}{2}\right)+w_0^2x_0\cos(wt-\delta)=\frac{f_0}{m}\cos(wt)$. The result of vectors addition is represented in the vector diagram on the Figure 2b.




a b

Figure 2 Vector diagrams of forced oscillations:

a - representation of members of the left side of equation $-w^2x_0\cos(wt-\delta)+2\gamma wx_0\cos\left(wt-\delta+\frac{\pi}{2}\right)+w_0^2x_0\cos(wt-\delta)=\frac{f_0}{m}\cos(wt)$ rotating vectors;

b - vector interpretation of equation $-w^2x_0\cos(wt-\delta)+2\gamma wx_0\cos\left(wt-\delta+\frac{\pi}{2}\right)+w_0^2x_0\cos(wt-\delta)=\frac{f_0}{m}\cos(wt)$.

We apply the Pythagorean Theorem, and then we have the following

$$(\frac{f_0}{m})^2 = (w_0^2 - w^2)^2 x_0^2 + 4\gamma^2 w^2 x_0^2$$

This equation makes it possible to determine the amplitude of forced oscillations, which is equal to

$$x_0 = \frac{f_0}{m\sqrt{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}}$$

The phase shift of forced oscillations, namely, the phase difference of the driving force $f(t) = f_0 \cos(wt)$ and displacement $x(t) = x_0 \cos(wt - \delta)$ can also be determined by the vector diagram on the Figure 4. From the Figure 4 we can conclude that driving force ahead offset by an angle δ, whose tangent is given by

$$tg(\delta) = \frac{2\gamma w}{w_0^2 - w^2}$$