Answer on Question #50819-Physics-Electromagnetism

A) A sphere of radius R carries a charge of volume charge density p = cr, where c is a constant and r denotes the distance from the center of the sphere. Calculate the total charge enclosed by the sphere and the electric field at points lying inside and outside the sphere.

B) A dielectric object is placed in an electric field. The object becomes polarized and a large number of atomic /molecular dipoles in the object align in the direction of the applied electric field. Derive an expression for the electric field produced by this polarized dielectric at a point outside the dielectric object.

C)Determine the magnitude of the pointing vector and the energy per unit time delivered to a wire of length L and cross-section A when electric current I flows in it.

Solution

A) The total charge enclosed by the sphere is

$$Q=\int_{0}^{2\pi} d\varphi \int_{0}^{\pi} \sin\theta\, d\theta \int_{0}^{R} (cr)r^2 dr = 2\pi\cdot 2\cdot\frac{cR^4}{4}=\pi c R^4.$$

The electric field outside a charged, spherical, conducting shell is the same as that generated when all the charge is concentrated at the center of the shell. Thus, the electric field at the distance r outside the sphere is

$$E=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}=\frac{c}{4\varepsilon_0}\frac{R^4}{r^2}.$$

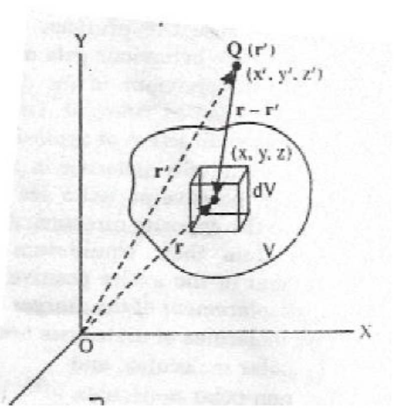
Applying Gauss's law at distances $r<R$ we get

$$4\pi r^2 E=\frac{1}{\varepsilon_0}\int_{0}^{2\pi} d\varphi \int_{0}^{\pi} \sin\theta\, d\theta \int_{0}^{R} (cr)r^2 dr=\frac{1}{\varepsilon_0}\pi c r^4.$$

The electric field at the distance r inside the sphere is

$$E=\frac{c}{4\varepsilon_0}r^2.$$

B) In the presence of an electric field, a dielectric becomes polarized and acquires a dipole moment P, per unit volume. The aim is to calculate field at point r [denoted by Q(r') in Fig. outside the dielectric. Let the volume of the dielectric is divided into small elementary volumes and



consider one such volume element dV of the dielectric. The potential due to dielectric body occupying volume dV as observed outside dV for single dipole potential can be expressed as,

$$\phi(r') = \frac{1}{4\pi\varepsilon_0}\int_V \frac{P(r).(r-r')}{|(r-r')|^3}dV$$

$$\phi(r') = \frac{1}{4\pi\varepsilon_0}\int_V \frac{P(r).\frac{(r-r')}{|(r-r')|}dV}{|(r-r')|^2}$$

$$\phi(r') = \frac{1}{4\pi\varepsilon_0}\int_V \frac{P(r).(r-r')^0}{|(r-r')|^2}dV$$

where (r-r')0 represents unit vector directed from Q towards dV. It is obviously equal to

$$\frac{(x'-x)i + (y'-y)j + (z'-z)k}{\sqrt{\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}}}$$

where x', y',z' are the Cartesian coordinates of point r' and (x,y,z)those at r. Further,

$$(r-r')^0 = -\left[i\frac{\partial}{\partial x}(r-r') + j\frac{\partial}{\partial x}(r-r') + k\frac{\partial}{\partial x}(r-r')\right]$$

$$P(r) = iP_x(x.y.z) + jP_y(x.y.z) + kP_z(x.y.z)$$

substituing these values in equation, we get

$$\phi(r') = \frac{1}{4\pi\varepsilon_0}\left\{\int_V \frac{P_x}{(r-r')^2}.\frac{\partial}{\partial x}(r-r')dV + \int_V \frac{P_y}{(r-r')^2}.\frac{\partial}{\partial y}(r-r')dV + \int_V \frac{P_z}{(r-r')^2}.\frac{\partial}{\partial z}(r-r')dV\right\}$$

$$= \phi_1 + \phi_2 + \phi_3$$

Now we shall consider the integrals separately. It is obvious that

$$-\frac{1}{|(r-r')|^2}.\frac{\partial(r-r')}{\partial x} = \frac{\partial}{\partial x}\frac{1}{|(r-r')|^2}$$

$$dV = dxdydz$$

Therefore

$$\phi_1 = \frac{1}{4\pi\varepsilon_0}\iiint P_x\frac{\partial}{\partial x}\frac{1}{|(r-r')|^2}dxdydz$$

After integration we get

$$\phi(r') = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\mathbf{P}.n}{|(r-r')|}dA - \frac{1}{4\pi\varepsilon_0}\int_V \frac{(-divP)dV}{|(r-r')|}$$

Equation obviously shows that the potential at point r' due to dielectric body is the same as if it were replaced by a system of bound charges in empty space. A part of these bound charges appears on the dielectric surfaces as a surface density óP' given by

$$\sigma_P' = \mathbf{P.n}$$

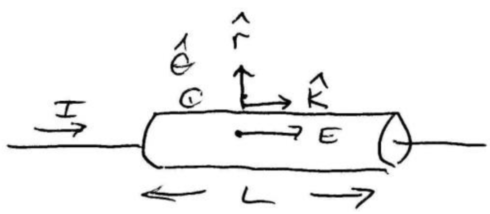
while the remaining bound charge appears throughout the volume V as a volume density rp' given as

$$\rho_P' = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right)$$

The intensity of electric field due to the polarized volume may be expressed as

$$E(r') = \frac{1}{4\pi\varepsilon_0}\left[\int_S \frac{\sigma_P'(r-r')}{|r-r'|^3}dA + \int_V \frac{\rho_P'(r-r')}{|r-r'|^3}dV\right]$$

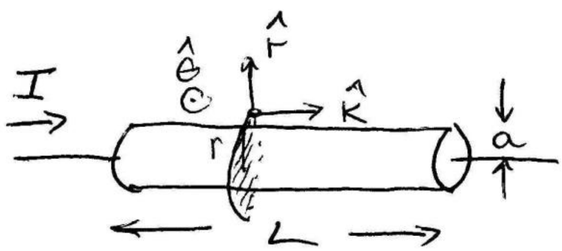
C) We choose unit vectors as shown in the figure below



There is an electric potential difference across the resistor that is equal to $\Delta V = IR$. The electric field is uniform in the resistor so $|\Delta V| = \left|\int \vec{E}d\vec{s}\right| = EL = IR$. Therefore the direction and magnitude of the electric field is given by

$$\vec{E} = \frac{IR}{L}\hat{k}.$$

We can use Ampere's Law to find the magnetic field surface of the resistor. Choose a circular loop as shown in the figure of radius $= a$.



Ampere's law is

$$\oint_{circle} \vec{B}d\vec{s} = \mu_0 \iint_{disk} \vec{J}\vec{n}\,da.$$

By symmetry,

$$\oint_{circle} \vec{B} d\vec{s} = B2\pi a.$$

The current through the disk is

$$\iint_{disk} \vec{J}\vec{n}\, da = I.$$

So Ampere's Law becomes

$$B2\pi a = \mu_0 I.$$

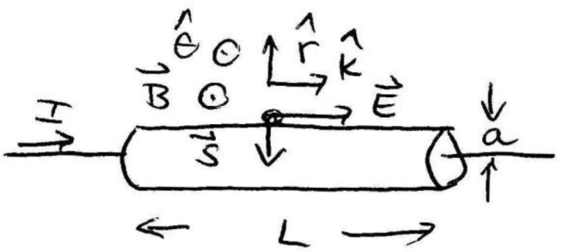
With our choice of unit vectors, the direction and magnitude of the magnetic field on the surface of the resistor is

$$\vec{B} = \frac{\mu_0 I}{2\pi a}\hat{\theta}.$$

The **Poynting vector** is given by

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{\left(\frac{IR}{L}\hat{k}\right) \times \left(\frac{\mu_0 I}{2\pi a}\hat{\theta}\right)}{\mu_0} = \frac{IR}{L}\frac{I}{2\pi a}(-\hat{r})$$

In order to calculate the power flow into the resistor we need to calculate the closed surface flux of the Poytning vector over the surface of the resistor.



Because the Poytning vector points radially inward we only calculate the flux over the cylindrical body of the resistor. Hence we set $r = a$ for Poynting vector and then integrate it over the outside cylindrical area of the resistor. For a closed surface we choose $\hat{n}_{out} = \hat{r}$ and so $\vec{S}\hat{r} = -|\vec{S}|$.

$$Power = \oiint \vec{S}\,\hat{n}_{out} da = \iint_{cylinder} \vec{S}\hat{r}\, da = -\iint_{cylinder} |\vec{S}|\, da = -|\vec{S}(r = a)|A.$$

The surface area in question is $= 2\pi aL$. Thus using our result for the magnitude of the Poynting vector we have that the power flow through the surface is

$$Power = -\left(\frac{IR}{L}\frac{I}{2\pi a}\right)2\pi aL = -I^2R.$$

The negative sign indicates that power is flowing into the resistor. We know that the rate that work is done by the electric field in the wire is I^2R so our answer makes because energy per sec flows in and is equal to the rate that work is done on the charges in the wire. This energy is dissipated in the form of random thermal motion due to collisions in the wire and eventually is transmitted to the motion of the air molecules surrounding the wire.

The energy per unit time delivered to a wire is

$$P = I^2 R.$$