Answer on Question #50746-Physics-Other

Obtain the temperature distribution $u(x,t)$ in a laterally insulated rod of length' L' if both ends on the rod are kept at (0 degree Celsius) and the initial temperature distribution in the bar is: $u(x,0) = 6\sin(\frac{\pi x}{L})$

Solution

The boundary conditions:

$$u(0,t) = u(L,t) = 0.$$

The initial condition:

$$u(x,0) = 6\sin\left(\frac{\pi x}{L}\right).$$

$$u = X(x)T(t).$$

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \to \frac{\dot{T}}{c^2 T} = \frac{X''}{X} = k.$$

$$X'' - kX = 0.$$

$$\dot{T} - kc^2 T = 0.$$

case1: $k = m^2 > 0.$

The general solution is:

$$X = Ae^{mx} + Be^{-mx}.$$

Applying the boundary conditions:

$$A = B = 0.$$

case2: $k = 0.$

The general solution is:

$$X = Ax + B.$$

Applying the boundary conditions:

$$A = B = 0.$$

case3: $k = -m^2 < 0.$

The general solution is:

$$X = A\sin mx + B\cos mx.$$

Applying the boundary conditions:

$$X(0) = B = 0.$$

$$X(L) = A\sin mL = 0 \to mL = n\pi \to m = \frac{n\pi}{L}.$$

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$\lambda_n = \frac{nc\pi}{L}.$$

We have:

$$\dot{T} + \left(\frac{nc\pi}{L}\right)^2 T = 0.$$

General solution is:

$$T_n = A_n e^{-\left(\frac{nc\pi}{L}\right)^2 t}.$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{nc\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

But

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = 6\sin\left(\frac{\pi x}{L}\right) \rightarrow A_n = 0 \; for \; n \neq 1, \;\; A_1 = 6.$$

Answer: $\boldsymbol{u(x,t) = 6e^{-\left(\frac{c\pi}{L}\right)^2 t} \sin\left(\frac{\pi x}{L}\right)}$.