Answer on Question 50676, Physics, Molecular Physics | Thermodynamics

Question:

What is Brownian motion? Give three examples of such a motion. Using Einstein's theory, obtain an expression for Einstein's formula for mean square displacement of a Brownian particle.

Answer:

Brownian motion is the random movement of microscopic particles suspended in a liquid or gas caused by their collision with the quick atoms or molecules in the surrounding medium. Three examples of such a motion: motion of pollen grains in water, diffusion of "holes" through a semiconductor, motion of smoke in a glass box.

Let us obtain an expression for Einstein's formula for mean square displacement of a Brownian particle. When Brownian particle is move the net force F acts on it. Also on particle acts the friction force f caused by the medium viscosity and directed opposite to the force F.

Let us suppose that the particle has a spherical shape of radius a. Then the friction force f can be expressed by the Stokes' law:

$$f = 6\pi\eta a v,$$

where, η is the dynamic viscosity, v is the velocity of the particle.

So, we can write the equation of motion of the particle:

$$m\ddot{r} = F - 6\pi\eta a\dot{r}, \quad (1)$$

where, m is a mass of the particle, r is the radius-vector of the particle relative to an arbitrary coordinate system, $\dot{r} = v$ is the velocity of the particle.

Let us consider the projection of the radius-vector on the axis X. Then the equation (1) looks like:

$$m\ddot{x} = F_x - 6\pi\eta a\dot{x}, \quad (2)$$

where, F_x is the projection of the net force F on the axis X.

We need to obtain the displacement of the Brownian particle x, which caused by the collisions with the molecules. Mean displacement of the particle $\bar{x}$ would be equal to zero, because the displacements of the particle with equal probability can have both positive and negative values. But the mean square displacement $\overline{x^2}$ is not equal to zero, and we can rewrite the equation (2) so that it includes the value of $\overline{x^2}$ (we multiply both sides of the equation on x):

$$mx\ddot{x} = F_x - 6\pi\eta a x\dot{x}. \quad (3)$$

Let us use the next identities:

$$x\ddot{x} = \frac{1}{2}\frac{d^2(x^2)}{dt^2} - \left(\frac{dx}{dt}\right)^2,\ x\dot{x} = \frac{1}{2}\frac{d(x^2)}{dt}.$$

Substituting this identities into the equation (3) we obtain:

$$\frac{m}{2}\frac{d^2(x^2)}{dt^2} - m\left(\frac{dx}{dt}\right)^2 = -3\pi\eta a\frac{d(x^2)}{dt} + xF_x.$$

This equality is valid for any particle and because of that it is also valid for mean values that includes in it, so we can write:

$$\frac{m}{2}\frac{d^2\left(\overline{x^2}\right)}{dt^2} - m\overline{\left(\frac{dx}{dt}\right)^2} = -3\pi\eta a\frac{d\left(\overline{x^2}\right)}{dt} + \overline{xF_x},$$

where $\overline{x^2}$ is the mean square displacement of the particle, $\overline{\left(\frac{dx}{dt}\right)^2}$ is the mean square velocity of the particle, the mean value of $\overline{xF_x}$ is equal to zero because for a large number of particles x and F_x equally takes both positive and negative values. Therefore, the equation (2) takes the next form:

$$\frac{m}{2}\frac{d^2\left(\overline{x^2}\right)}{dt^2} - m\overline{\left(\frac{dx}{dt}\right)^2} = -3\pi\eta a\frac{d\left(\overline{x^2}\right)}{dt}. \quad (4)$$

Because the motion of the particles is quite chaotic, then the mean squares velocity projections on all three coordinate axes must be equal to each other:

$$\overline{\left(\frac{dx}{dt}\right)^2} = \overline{\left(\frac{dy}{dt}\right)^2} = \overline{\left(\frac{dz}{dt}\right)^2}.$$

Also obvious, that the sum of this values must be equal to the mean square velocity of the particles $\overline{v^2}$:

$$\overline{\left(\frac{dx}{dt}\right)^2} + \overline{\left(\frac{dy}{dt}\right)^2} + \overline{\left(\frac{dz}{dt}\right)^2} = \overline{v^2}.$$

Therefore, we obtain:

$$\overline{\left(\frac{dx}{dt}\right)^2} = \frac{1}{3}\overline{v^2},$$

$$m\overline{\left(\frac{dx}{dt}\right)^2} = \frac{1}{3}m\overline{v^2} = \frac{2}{3}\frac{m\overline{v^2}}{2}.$$

Because the average kinetic energy of the Brownian particle must be equal to the average kinetic energy of the molecules of liquid (or gas), we can write:

$$\frac{m\overline{v^2}}{2} = \frac{3}{2}kT,$$

$$m\overline{\left(\frac{dx}{dt}\right)^2} = \frac{2}{3}\frac{m\overline{v^2}}{2} = kT. \quad (5)$$

Substituting the equation (5) into the equation (4) we obtain:

$$\frac{m}{2}\frac{d^2\left(\overline{x^2}\right)}{dt^2} - kT = -3\pi\eta a\frac{d\left(\overline{x^2}\right)}{dt}.$$

This equation can be easily integrated. Let us denote $\frac{d\left(\overline{x^2}\right)}{dt} = Z$ and rewrite the equation:

$$\frac{m}{2}\frac{dZ}{dt} - kT = -3\pi\eta a Z.$$

After separation of variables we obtain:

$$\frac{dZ}{Z - \frac{kT}{3\pi\eta a}} = -\frac{6\pi\eta a}{m}dt.$$

Integrating the left-side of the equation within the limits from 0 to Z and the the right-side of the equation within the limits from 0 to t we get:

$$\int_0^Z \frac{dZ}{Z - \frac{kT}{3\pi\eta a}} = -\int_0^t \frac{6\pi\eta a}{m}dt$$

$$\ln\left(Z - \frac{kT}{3\pi\eta a}\right) - \ln\left(-\frac{kT}{3\pi\eta a}\right) = -\frac{6\pi\eta a}{m}t.$$

From this equation we can find Z:

$$Z = \frac{kT}{3\pi\eta a}\left(1 - e^{-\frac{6\pi\eta a}{m}t}\right) = \frac{d\left(\overline{x^2}\right)}{dt}.$$

The value of $e^{-\frac{6\pi\eta a}{m}t}$ is negligible, so we can write:

$$\frac{d}{dt}\left(\overline{x^2}\right) = \frac{kT}{3\pi\eta a}. \text{ (6)}$$

For the finite time intervals Δt and appropriate displacements $\Delta\overline{x^2}$ the equation (6) takes form:

$$\frac{\Delta\overline{x^2}}{\Delta t} = \frac{kT}{3\pi\eta a}.$$

Finally, we obtain an expression for Einstein's formula for mean square displacement of a Brownian particle:

$$\Delta\overline{x^2} = \frac{kT}{3\pi\eta a}\Delta t.$$