Answer on Question #50642-Physics-Molecular Physics-Thermodynamics

What is Bose-Einstein condensation? Show that Bose-Einstein condensation temperature is given by

$$T_c = \frac{h^2}{2\pi m k_B} \cdot \left[\frac{N}{2.612V}\right]^{\frac{2}{3}}$$

Answer

A Bose-Einstein condensate is a rare state (or phase) of matter in which a large percentage of bosons collapse into their lowest quantum state, allowing quantum effects to be observed on a macroscopic scale. The bosons collapse into this state in circumstances of extremely low temperature, near the value of absolute zero.

The BEC critical temperature for a uniform 3D system is given by the condition that all the particles accommodated in excited single particles state (except for a ground state) when $\mu = u_0 = 0$ are equal to the total number of particles in the system:

$$N = \int_0^{\infty} g(u)\overline{n_u}du.$$

where $g(u)du$ is the density of states in terms of the energy density:

$$g(u)du = \frac{V}{4\pi^2}\left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \sqrt{u}du.$$

Thus

$$N = \frac{V}{4\pi^2}\left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \int_0^{\infty} \sqrt{u}\frac{du}{e^{\frac{u}{k_BT}} - 1}.$$

We can evaluate the energy integral using the relation $\frac{1}{e^x-1} = \sum_{n=1}^{\infty} e^{-nx}$, where $x = \frac{u}{k_BT}$:

$$\int_0^{\infty} \sqrt{u}\frac{du}{e^{\frac{u}{k_BT}} - 1} = (k_BT)^{\frac{3}{2}} \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nx}\sqrt{x}dx = (k_BT)^{\frac{3}{2}} \sum_{n=1}^{\infty} n^{-\frac{3}{2}} \int_0^{\infty} e^{-t}\sqrt{t}dt = (k_BT)^{\frac{3}{2}}\zeta\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)$$

where $\zeta\left(\frac{3}{2}\right) = \sum_{n=1}^{\infty} n^{-\frac{3}{2}}$ and $\Gamma\left(\frac{3}{2}\right) = \int_0^{\infty} e^{-t}\sqrt{t}dt.$

Finally, we obtain the BEC critical density for a uniform 3D system:

$$n_c = \frac{N_c}{V} = \frac{1}{4\pi^2}\left(\frac{2mk_BT}{h^2}\right)^{\frac{3}{2}} \zeta\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right).$$

We know that

$$\zeta\left(\frac{3}{2}\right) \cong 2.612; \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

For a given density $\frac{N}{V}$, the previous relation , written in terms of the temperature, defines the critical temperature T_c:

$$T_c = \frac{h^2}{2\pi m k_B} \cdot \left[\frac{N}{2.612V}\right]^{\frac{2}{3}}.$$