Answer on Question#47581 - Physics - Quantum Mechanics

Consider a charged particle bound in the harmonic oscillator potential $V(x) = 1/2m\omega^2x^2$. A weak electric field E is applied to the system such that the potential energy is shifted by an amount $H' = -qEx$.

(a) Calculate the energy levels of the perturbed system to second order in the small perturbation.

(b) Show that the perturbed system can be solved exactly by completing the square in the Hamiltonian. Compare the exact energies with the perturbation results found in (a).

Solution:

(a) The Hamiltonian for a simple harmonic oscillator in one dimension is

$$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2$$

and the (unperturbed) energy levels are

$$E_n^{(0)} = \left(n + \frac{1}{2}\right)\hbar\omega$$

as can be recalled from elementary quantum mechanics. For the matrix elements of x holds

$$\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1}\right).$$

The perturbation expansion for the energy levels is given by

$$E_n = E_n^{(0)} + \langle n^{(0)}|H'|n^{(0)}\rangle + \sum_{m\neq n}\frac{\left|\langle n^{(0)}|H'|m^{(0)}\rangle\right|^2}{E_n^{(0)} - E_m^{(0)}} + \cdots$$

To lowest non-vanishing order, this read (notice that $\langle n|H'|n\rangle = 0$)

$$E_n = E_n^{(0)} + (qE)^2\left(\frac{|\langle n|x|n-1\rangle|^2}{\left(n+\frac{1}{2}\right)\hbar\omega - \left(n-1+\frac{1}{2}\right)\hbar\omega} + \frac{|\langle n|x|n+1\rangle|^2}{\left(n+\frac{1}{2}\right)\hbar\omega - \left(n+1+\frac{1}{2}\right)\hbar\omega}\right)$$

$$= \left(n+\frac{1}{2}\right)\hbar\omega + \frac{q^2E^2}{\hbar\omega}\left(\frac{\hbar n}{2m\omega} - \frac{\hbar(n+1)}{2m\omega}\right) =$$

$$= \left(n+\frac{1}{2}\right)\hbar\omega - \frac{q^2E^2}{2m\omega^2}.$$

(b) We can solve this problem also in closed form. By completing the square we obtain

$$H_0 - qEx = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}\omega^2x^2 - qEx =$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2\left(x - \frac{qE}{m\omega^2}\right)^2 - \frac{q^2E^2}{2m\omega^2}$$

If we denote $y = x - qE/(m\omega^2)$, the Schrodinger equation to be solved is then

$$-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}\psi_n + \frac{1}{2}m\omega^2y^2\psi_n = \left(E_n + \frac{q^2E^2}{2m\omega^2}\right)\psi_n.$$

This is nothing but the Schrodinger equation for a simple harmonic oscillator, and therefore

$$\left(n+\frac{1}{2}\right)\hbar\omega = E_n + \frac{q^2E^2}{2m\omega^2}$$

from which we obtain

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{q^2E^2}{2m\omega^2}.$$

This result agrees with the energy shift we calculated using perturbation theory to lowest non-vanishing order.

Answer:

(a) $E_n = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{q^2E^2}{2m\omega^2}$

(b) $E_n = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{q^2E^2}{2m\omega^2}$