Answer on Question #47226, Physics, Mechanics | Kinematics | Dynamics

A rotating frame of reference is a special case of a non-inertial reference frame that is rotating relative to an inertial reference frame. An everyday example of a rotating reference frame is the surface of the Earth

Time derivatives in the two frames[edit]

Introduce the unit vectors $\hat{\boldsymbol{\imath}},\ \hat{\boldsymbol{\jmath}},\ \hat{\boldsymbol{k}}$ representing standard unit basis vectors in the rotating frame. As they rotate they will remain normalized. If we let them rotate at the speed of Ω about an axis $\boldsymbol{\Omega}$ then each unit vector $\hat{\boldsymbol{u}}$ of the rotating coordinate system abides by the following equation:

$$\frac{d}{dt}\hat{\boldsymbol{u}} = \boldsymbol{\Omega} \times \hat{\boldsymbol{u}}\ .$$

Then if we have a vector function $\boldsymbol{f}$,

$$\boldsymbol{f}(t) = f_x(t)\hat{\boldsymbol{\imath}} + f_y(t)\hat{\boldsymbol{\jmath}} + f_z(t)\hat{\boldsymbol{k}}\ ,$$

and we want to examine its first derivative we have

$$\begin{aligned}\frac{d}{dt}\boldsymbol{f} &= \frac{df_x}{dt}\hat{\boldsymbol{\imath}} + \frac{d\hat{\boldsymbol{\imath}}}{dt}f_x + \frac{df_y}{dt}\hat{\boldsymbol{\jmath}} + \frac{d\hat{\boldsymbol{\jmath}}}{dt}f_y + \frac{df_z}{dt}\hat{\boldsymbol{k}} + \frac{d\hat{\boldsymbol{k}}}{dt}f_z \\ &= \frac{df_x}{dt}\hat{\boldsymbol{\imath}} + \frac{df_y}{dt}\hat{\boldsymbol{\jmath}} + \frac{df_z}{dt}\hat{\boldsymbol{k}} + [\boldsymbol{\Omega}\times(f_x\hat{\boldsymbol{\imath}} + f_y\hat{\boldsymbol{\jmath}} + f_z\hat{\boldsymbol{k}})] \\ &= \left(\frac{d\boldsymbol{f}}{dt}\right)_r + \boldsymbol{\Omega}\times\boldsymbol{f}(t)\ ,\end{aligned}$$

where $\left(\frac{d\boldsymbol{f}}{dt}\right)_r$ is the rate of change of $\boldsymbol{f}$ as observed in the rotating coordinate system. As a shorthand the differentiation is expressed as:

$$\frac{d}{dt}\boldsymbol{f} = \left[\left(\frac{d}{dt}\right)_r + \boldsymbol{\Omega}\times\right]\boldsymbol{f}\ .$$

This result is also known as the Transport Theorem in analytical dynamics and is also sometimes referred to as the Basic Kinematic Equation.

Relation between velocities in the two frames

A velocity of an object is the time-derivative of the object's position, or

$$\mathbf{v} \stackrel{\text{def}}{=} \frac{d\mathbf{r}}{dt}$$

The time derivative of a position $\boldsymbol{r}(t)$ in a rotating reference frame has two components, one from the explicit time dependence due to motion of the particle itself, and another from the frame's own rotation. Applying the result of the previous subsection to the displacement $\boldsymbol{r}(t)$, the velocities in the two reference frames are related by the equation

$$\mathbf{v_i} \stackrel{\text{def}}{=} \frac{d\mathbf{r}}{dt} = \left(\frac{d\mathbf{r}}{dt}\right)_{\mathrm{r}} + \boldsymbol{\Omega}\times\mathbf{r} = \mathbf{v_r} + \boldsymbol{\Omega}\times\mathbf{r}\ ,$$

where subscript i means the inertial frame of reference, and r means the rotating frame of reference.

Relation between accelerations in the two frames

Acceleration is the second time derivative of position, or the first time derivative of velocity

$$\mathbf{a}_{\mathrm{i}} \stackrel{\mathrm{def}}{=} \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\mathrm{i}} = \left(\frac{d\mathbf{v}}{dt}\right)_{\mathrm{i}} = \left[\left(\frac{d}{dt}\right)_{\mathrm{r}} + \boldsymbol{\Omega}\times\right]\left[\left(\frac{d\mathbf{r}}{dt}\right)_{\mathrm{r}} + \boldsymbol{\Omega}\times\mathbf{r}\right] ,$$

where subscript i means the inertial frame of reference. Carrying out the differentiations and re-arranging some terms yields the acceleration in the rotating reference frame

$$\mathbf{a}_{\mathrm{r}} = \mathbf{a}_{\mathrm{i}} - 2\boldsymbol{\Omega}\times\mathbf{v}_{\mathrm{r}} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) - \frac{d\boldsymbol{\Omega}}{dt}\times\mathbf{r}$$

where $\mathbf{a}_{\mathrm{r}} \stackrel{\mathrm{def}}{=} \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\mathrm{r}}$ is the apparent acceleration in the rotating reference frame, the term $-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$ represents centrifugal acceleration, and the term $-2\boldsymbol{\Omega}\times\mathbf{v}_{\mathrm{r}}$ is the coriolis acceleration.