Answer on Question #44795, Physics, Mechanics | Kinematics | Dynamics

Question:

Establish the differential equation for damped harmonic oscillator and obtain its solution. Show that the damped oscillator will exhibit non-oscillatory behaviour if the damping is heavy.

Answer:

An ideal mass–spring–damper system with mass m, spring constant k and viscous damper of damping coefficient c is subject to an oscillatory force:

$$F_s = -kx$$

and a damping force

$$F_d = -cv = -c\frac{dx}{dt} = -c\dot{x}$$

Treating the mass as a free body and applying Newton's second law, the total force F_{tot} on the body is:

$$F_{tot} = -kx - c\dot{x} = m\ddot{x}$$

This differential equation may be rearranged into

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

The following parameters are then defined:

$$\omega_0 = \sqrt{\frac{k}{m}}, \zeta = \frac{c}{2\sqrt{mk}}$$

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0$$

Continuing, we can solve the equation by assuming a solution x such that:

$$x = e^{\gamma t}$$

where the parameter γ is, in general, a complex number.

Substituting this assumed solution back into the differential equation gives:

$$\gamma^2 + 2\zeta\omega_0\gamma + \omega_0^2 x = 0$$

Solving the characteristic equation will give two roots:

$$\gamma_\pm = -\zeta\omega_0 \pm \sqrt{(\zeta\omega_0)^2 - \omega_0^4}$$

The solution to the differential equation is thus:

$$x(t) = Ae^{\gamma_+ t} + Be^{\gamma_- t}$$

where A and B are determined by the initial conditions of the system:

$$A = x(0) + \frac{\gamma_+ x(0) - \dot{x}(0)}{\gamma_- - \gamma_+}$$

$$B = -\frac{\gamma_+ x(0) - \dot{x}(0)}{\gamma_- - \gamma_+}$$

When $\zeta > 1$, the system is over-damped and there are two different real roots.

The solution to the motion equation is:

$$x(t) = Ae^{\gamma_+ t} + Be^{\gamma_- t}$$

with real γ_+ and γ_- therefore it is non-oscillatory behavior.