Answer on Question #40595, Physics, Acoustics

What do you understand by wave impedance? Derive an expression for the impedance offered to sound waves propagating in a gaseous medium.

Solution:

Impedance. A very useful concept for understanding the flow of energy is called the impedance. Energy flow, or the rate at which energy is being used, is called power. If you push on a moving object with a force F and it moves with velocity v then the rate of energy transfer (the power) is equal to the force multiplied by the velocity.

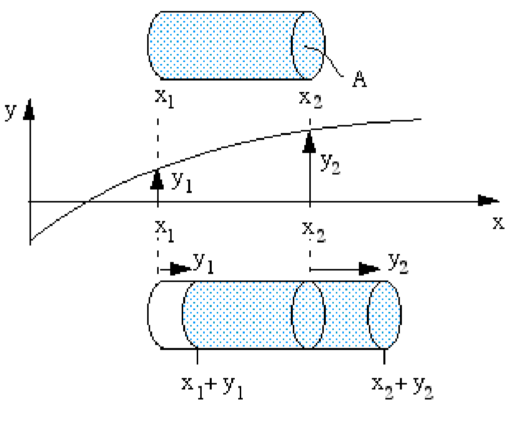
$$P = Fv$$

Of course, just how fast the object moves will depend on the force you apply. The two are not independent. For a harmonic force, the ratio of force to velocity tells you how hard it is to move the object – this depends both on how much inertia the object has and on how strong the restoring force is.

$$Z = \frac{Force}{velocity}$$

Wave impedance. As a sound wave propagates along, each piece of the material transfers the energy to the next in a kind of "bucket brigade" fashion. From a physics point of view, as long as the impedance does not change, the propagation of the wave is not disturbed. This is what makes the impedance important. At an interface between two media (for example when sound hits a wall) you often what to know how much, if any, of the wave energy is going to be reflected. If the impedance of the wall is the same as the impedance of the air then there will be no reflection of power. If the impedance of the wall is either greater or lesser than the air, then there will be significant reflection. If the impedance changes slowly, then the wave will not reflect either.

Sound waves propagating in a gaseous medium.



The wave we analyse here is sound in an elastic medium, such as air. Let's consider a one dimensional wave travelling in the x direction. We'll consider a section with cross section A. (We might picture a wave travelling through a pipe, and neglect the viscous and thermal interactions at the wall.)

We'll consider the motion of an element of our medium, whose position, when there is no sound wave disturbance, is between x_1 and x_2. We'll make the distance between x_1 and x_2 much less than a wavelength of sound, and later shall take the limit of very small distance.

So the undisturbed volume of this element is

$$V = A(x_2 - x_1)$$

and, using the density ρ and cross section A, its mass is

$$m = \rho V = \rho A(x_2 - x_1) \quad (1).$$

At any time t, y is the average displacement of the molecules in the x direction. (We write 'average', because of course in a gas, the individual molecules have their own random motion superimposed on the average or bulk motion that we are analysing.) So the illustration shows that, at some time t,

- the medium whose equilibrium position is at x_1 is moved to $x_1 + y_1$ and
- the medium whose equilibrium position is at x_2 is moved to $x_2 + y_2$.

he new volume of our element of air is

$$V + \delta V = A(x2 + y2 - x1 - y1) \quad \text{and, using (1), we have}$$

$$\delta V = A(y2 - y1).$$

When a medium is compressed, its pressure rises. The ratio between the pressure increase p and the proportional volume reduction is called the volumetric modulus of elasticity κ, defined by

$$p = -\kappa \frac{\delta V}{V}$$

So the pressure in our element, when it is displaced as shown in our diagram, is

$$p = -\kappa \frac{\delta V}{V} = -\kappa \frac{A(y_2 - y_1)}{A(x_2 - x_1)} = -\kappa \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

So the relation between displacement y and pressure p:

$$p = -\kappa \frac{\partial y}{\partial x} \qquad (2)$$

Now we apply Newton's second law. The absolute pressure at x_1 is $P_{Atm} + p_1$, where P_{Atm} is undisturbed atmospheric pressure. So the pressure on the left, acting on area A, exerts a force to the right of $(P_{Atm} + p_1)A$. Similarly, at x_2, the force on the right, acting to the left, is $(P_{Atm} + p_2)A$. So the net force to the right is

Let's take the average displacement of our element as y (where $y_1 < y < y_2$). So, taking the second derivative with respect to time, its acceleration to the right is

$$a = \frac{\partial^2 y}{\partial t^2}$$

Equation (1) gives the mass and the two equations above give F and a, so F = ma becomes

$$(p_2 - p_1)A = \rho A(x_2 - x_1)\frac{\partial^2 y}{\partial t^2}$$

We rearrange, reverse the order of p_1 and p_2 and cancel A to give

$$\frac{(p_2 - p_1)}{(x_2 - x_1)} = \frac{\partial p}{\partial x} = -\rho \frac{\partial^2 y}{\partial t^2}$$

Combining this with the equation (2) we have

$$\frac{\partial^2 y}{\partial t^2} = \frac{\kappa}{\rho}\frac{\partial^2 y}{\partial x^2}$$

This is the one dimensional wave equation.

We seek a solution

$$y = y_m \sin(kx - \omega t)$$

this is a solution provided that $(\omega/k)^2 = \kappa/\rho$. Now ω/k is the wave speed so we have, for the speed of sound:

$$v = \sqrt{\frac{\kappa}{\rho}}$$

Although we imagined that our element was air, we have so far only assumed that it is elastic. So this equation is true for longitudinal waves in any elastic medium.

The speed of sound in an ideal gas is

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

where γ is the adiabatic constant, the pressure is P.

The particle velocity is

$$u = \frac{\partial y}{\partial t} = -y_m \omega \cos(kx - \omega t)$$

Using equation (2), we have

$$p = -\kappa \frac{\partial y}{\partial x} = -\kappa y_m k \cos(kx - \omega t)$$

The specific acoustic impedance z of a medium is defined by p/u, so, using the equations above

$$z = \kappa k/\omega.$$

Above, w e had $\kappa/\rho = v^2$, above so $\kappa = \rho v^2$. We also had $\omega/k = v$. Substituting these gives the specific acoustic impedance

$$z = \rho v.$$

We define acoustic impedance Z and specific acoustic impedance z thus:

$$Z = p/U \quad \text{and} \quad z = p/u$$

In all cases, 'acoustic' refers to the oscillating component. With this proviso, we can say that

acoustic impedance = pressure/flow

and specific acoustic impedance = pressure/velocity

The specific acoustic impedance, z, is an intensive property of a medium. We can specify the z of air or of water. The acoustic impedance Z is the property of a particular geometry and medium: we can discuss for example the Z of a particular duct filled with air. Usually, Z varies strongly when the frequency changes. The acoustic impedance at a particular frequency indicates how much sound pressure is generated by a given acoustic flow at that frequency.

Let's now consider a one dimensional wave passing through an aperture with area A. The volume flow U is the volume passing per second through the aperture. If the flow moves a distance dy = udt, then the volume passing through is A·dy, so

$$U = \frac{dV}{dt} = A\frac{dy}{dt} = Au$$

The acoustic impedance Z is the ratio of sound pressure to volume flow, so provided that the wave is only one dimensional, we should have

$$Z = \frac{p}{U} = \frac{p}{A \cdot u} = \frac{z}{A} = \frac{\rho v}{A}$$