Answer on Question #40591, Physics, Mechanics | Kinematics | Dynamics

Establish the differential equation for damped harmonic oscillator and obtain its solution. Show that the damped oscillator will exhibit non-oscillatory behavior if the damping is heavy.

Solution

An ideal mass–spring–damper system with mass m, spring constant k and viscous damper of damping coefficient c is subject to an oscillatory force

$$F_{\mathrm{osc}} = -kx,$$

and a damping force

$$F_{\mathrm{d}} = -cv = -c\frac{dx}{dt}.$$

Applying Newton's second law, the total force F on the body is

$$F = ma = m\frac{d^2x}{dt^2},$$

where a is the acceleration of the mass and x is the displacement of the mass relative to a fixed point of reference.

Since $F = F_{\mathrm{osc}} + F_d$

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}.$$

This differential equation may be rearranged into

$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$ or $\left\{\frac{d^2}{dt^2} + \frac{c}{m}\frac{d}{dt} + \frac{k}{m}\right\}x = 0.$

In this differential equation $\frac{d}{dt} = D$ is differential operator. So

$$\left\{D^2 + \frac{c}{m}D + \frac{k}{m}\right\}x = 0.$$

Let's complete the square

$$\left\{D^2 + \frac{c}{m}D + \frac{k}{m}\right\}x = \left\{\left(D + \frac{c}{2m}\right)^2 + \left(\frac{k}{m} - \left(\frac{c}{2m}\right)^2\right)\right\}x = 0.$$

We can set

$$\omega^2 = \frac{k}{m} - \left(\frac{c}{2m}\right)^2.$$

We are assuming here that $\omega^2 \geq 0$.

Now our differential equation reads

$\left\{\left(\frac{d}{dt} + \frac{c}{2m}\right)^2 + \omega^2\right\}x = 0$ or $\left(\frac{d}{dt} + \frac{c}{2m}\right)^2 x = -\omega^2 x.$

Let's take the square root of both sides of this equation

$$\left(\frac{d}{dt}+\frac{c}{2m}\right)x=\pm i\omega x.$$

Now we have two first order equations to solve:

$$\frac{dx}{dt}=\left(-\frac{c}{2m}\pm i\omega\right)x,$$

which have the solution

$$x=e^{-\frac{c}{2m}t}(A\sin\omega t+B\cos\omega t).$$

It is the general form of the solution representing damped oscillations with damped frequency

$\omega=\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}$.

<u>The overdamped case </u>(<u>damping is heavy</u>) occurs when $\omega_0=\frac{k}{m}<\frac{c}{2m}$. Now the system doesn't oscillate at all; the motion simply dies away. This is characterised by a solution which decays exponentially.

Let’s rewrite our equation

$$\left\{\left(\frac{d}{dt}+\frac{c}{2m}\right)^2-\omega^2\right\}x=0$$

where $\omega^2=\left(\frac{c}{2m}\right)^2-\frac{k}{m}>0.$

Then upon square rooting our equation we obtain

$$\left(\frac{d}{dt}+\frac{c}{2m}\right)x=\pm\omega x,$$

which have a solution

$$x=Ce^{(-\frac{c}{2m}+\omega)t}+De^{(-\frac{c}{2m}-\omega)t}.$$

This solution represents a damped motion without oscillations.