Answer on Question #39278, Physics, Mechanics | Kinematics | Dynamics

Establish the differential equation for damped harmonic oscillator and obtain its solution. Show that the damped oscillator will exhibit non-oscillatory behavior if the damping is heavy.

Solution:

Newton's Law for a spring system with linear damping reads

$$-kx - bv = ma$$

for a block of mass *m* attached to a spring of constant *k* with damping coefficient *b*.

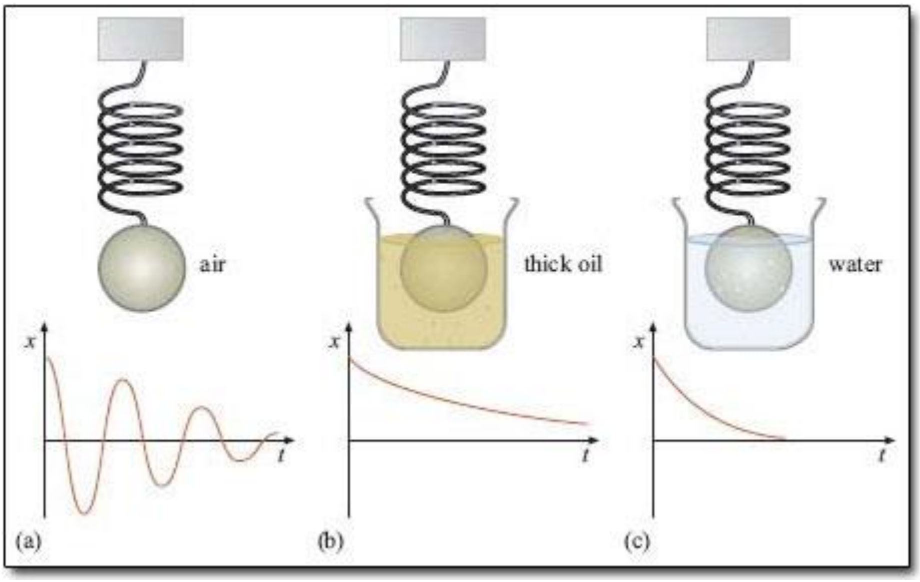

Figure: Plots of displacement vs time for the mass-spring system: (a) underdamped – mass in air; (b) overdamped – mass in thick oil; (c) critically damped – mass in water

Using the definitions of velocity and acceleration we can write this as the differential equation

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\left\{\frac{d^2}{dt^2} + \frac{b}{m}\frac{d}{dt} + \frac{k}{m}\right\}x = 0$$

We can think of the expression on the left hand side as a polynomial in the variable d/dt. We proceed by making the substitution y = d/dt and then completing the square

$$y^2 + \frac{b}{m}y + \frac{k}{m} = y^2 + 2\left(\frac{b}{2m}\right) + \left(\frac{b}{2m}\right)^2 - \left(\frac{b}{2m}\right)^2 + \frac{k}{m} = \left(y + \frac{b}{2m}\right)^2 + \frac{k}{m} - \left(\frac{b}{2m}\right)^2$$

So now our differential equation reads

$$\left\{\left(\frac{d}{dt} + \frac{b}{2m}\right)^2 + \omega^2\right\}x = 0,$$

where we have set

$$\omega^2 = \frac{k}{m} - \left(\frac{b}{2m}\right)^2.$$

We are assuming here that $\omega^2 > 0$.

Now we just move one term to the other side to get

$$\left(\frac{d}{dt}+\frac{b}{2m}\right)^2 x = -\omega^2 x,$$

and we take the square root of this expression to get

$$\left(\frac{d}{dt}+\frac{b}{2m}\right) x = \pm i\omega x.$$

Note that we now have two first order equations to solve (one for each sign).

We seek solutions to the equations

$$\frac{dx}{dt} = \left(-\frac{b}{2m} \pm i\omega\right) x,$$

which have the obvious solutions

$$x = \exp\left(-\frac{b}{2m} \pm i\omega\right) t = \exp\left(-\frac{b}{2m} t\right) \exp(\pm i\omega t) = \exp\left(-\frac{b}{2m} t\right) (\cos(\omega t) \pm i\sin(\omega t))$$

Thus our two solutions are (using Euler's formula)

$$x_1 = A_1 \exp\left(-\frac{b}{2m} t\right) (\cos(\omega t) + i\sin(\omega t))$$

$$x_2 = A_2 \exp\left(-\frac{b}{2m} t\right) (\cos(\omega t) - i\sin(\omega t))$$

and our total solution (x1 + x2) can be written

$$x = \exp\left(-\frac{b}{2m} t\right) \left((A_1 + A_2) \cos(\omega t) + i(A_1 - A_2)\sin(\omega t)\right).$$

Now, we need to choose A1 and A2 so that we get a real-valued solution, that is A1 + A2 is real, and A1 − A2 is imaginary.

This condition has the effect of taking us from four unknown quantities (the real and imaginary part of each A) to just two, which is the appropriate number for a second order equation. Our solution is now

$$x = \exp\left(-\frac{b}{2m} t\right) \left(B\cos(\omega t) + C\sin(\omega t)\right),$$

which is the general form of the solution representing damped oscillations, and we have

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

2) The overdamped case (damping is heavy) occurs when $\omega_0 < b/2m$. Now the system doesn't oscillate at all; the motion simply dies away. This is characterised by a solution which decays exponentially.

Then we rewrite our equation as

$$\left\{\left(\frac{d}{dt}+\frac{b}{2m}\right)^2 - \omega^2\right\} x = 0,$$

where we now have set

$$\omega^2 = \left(\frac{b}{2m}\right)^2 - \frac{k}{m} > 0.$$

Then upon square rooting our equation we obtain

$$\left(\frac{d}{dt}+\frac{b}{2m}\right) x = \pm \omega x,$$

which is a real equation. The differential equation to solve is now

$$\frac{dx}{dt} = \left(-\frac{b}{2m} \pm \omega\right) x,$$

which has the solutions

$$x_1 = A_1 \exp\left(-\frac{b}{2m} + \omega\right) t,$$
$$x_2 = A_2 \exp\left(-\frac{b}{2m} - \omega\right) t,$$

both representing a damped motion **without oscillations**.