Answer on Question 39277, Physics, Other

The equation of motion of a forced weakly damped oscillator is $\ddot{x}+2\lambda\dot{x}+\omega_0^2 x=\frac{f}{m}\cos\gamma t$, where $\ddot{x}$ is the acceleration, $2\lambda\dot{x}$ takes damping into account, and $\frac{f}{m}\cos\gamma t$ is the external periodic force.

Let us first obtain the solution of homogenous equation. $\ddot{x}+2\lambda\dot{x}+\omega_0^2 x=0$. Let us look for a solution in form $x=e^{rt}$. Plugging this into equation, obtain $r^2+2\lambda r+\omega_0^2=0$. General solution is hence $x=a_1 e^{r_1 t}+a_2 e^{r_2 t}$, where $r_{12}=-\lambda\pm\sqrt{\lambda^2-\omega_0^2}$. If $\lambda<\omega_0$, then we have two complex solutions, which are complex conjugate one to each other. Thus, solution might be written as $x=a e^{-\lambda t}\cos(\omega t+\alpha)$, where $\omega=\sqrt{(\omega_0^2-\lambda^2)}$.

Knowing the homogenous solution, one has to find the particular solution of $\ddot{x}+2\lambda\dot{x}+\omega_0^2 x=\frac{f}{m}\cos\gamma t$. Seek for it in form $x_p(t)=A\cos\gamma t+B\sin\gamma t$. Plugging in into equation, obtain $A=\frac{-f}{m}\frac{(\gamma^2-\omega_0^2)}{[(\omega_0^2-\gamma^2)+4\lambda^2\gamma^2]}$ and $B=\frac{f}{m}\frac{2\gamma\lambda}{[(\omega_0^2-\gamma^2)+4\lambda^2\gamma^2]}$.

Particular solution $x_p(t)=A\cos\gamma t+B\sin\gamma t$ might be then rewritten as $\sqrt{A^2+B^2}(\cos\delta\cos\gamma t-\sin\delta\sin\gamma t)$, where $\cos\delta=\frac{A}{\sqrt{A^2+B^2}}$, $\sin\delta=\frac{-B}{\sqrt{A^2+B^2}}$, thus $\tan\delta=\frac{-B}{A}=\frac{2\gamma\lambda}{\gamma^2-\omega_0^2}$.

Hence, $x_p(t)=C\cos(\gamma t+\delta)$, where $C=\frac{f}{m\sqrt{(\omega_0^2-\gamma^2)^2+4\lambda^2\gamma^2}}$.

Finally, solution of equation is $x=a e^{-\lambda t}\cos(\omega t+\alpha)+C\cos(\gamma t+\delta)$.