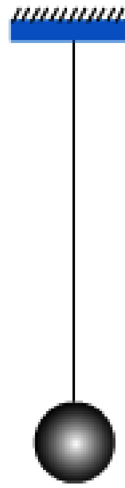
Answer on Question#38189 – Physics – Other

Prove that the motion of a simple pendulum is simple harmonic motion. Also show that the time period of simple pendulum of very large length is independent of length

Solution:

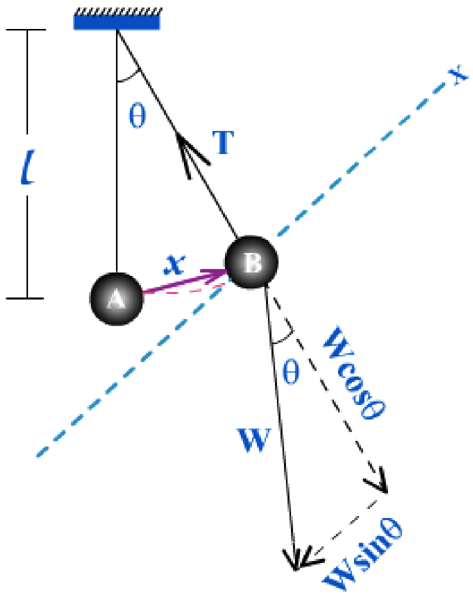
#1

Simple pendulum consists of a heavy mass particle suspended by a light, flexible and in-extensible string.



The motion of the bob of simple pendulum simple harmonic motion if it is given small displacement. In order to prove this fact consider a simple pendulum having a bob of mass 'm' and the length of pendulum is 'l'. Assuming that the mass of the string os pendulum is negligible. When the pendulum is at rest at position 'A', the only force acting is its weight and tension in the string. When it is displaced from its mean position to another new position say 'B' and released, it vibrates to and fro around its mean position.

Suppose that at this instant the bob is at point 'B' as shown below:



Forces acting on the bob:
Weight of the bob acting vertically downward.
Tension in the string (T) acting along the string.

The weight of the bob can be resolved into two rectangular components:

a. Wcosθ along the string.
b. Wsinθ perpendicular to string.

Since there is no motion along the string, therefore, the component Wcosθ must balance tension (T)
i.e.

$$W\cos\theta = T$$

This shows that only Wsinθ is the net force which is responsible for the acceleration in the bob of pendulum.
According to Newton's second law of motion Wsinθ will be equal to $m \times a$
i.e.

$$W\sin\theta = ma$$

Since Wsinθ is towards the mean position, therefore, it must have a negative sign.
i.e.

$$W\sin\theta = -ma$$

But W = mg:

$$mg\sin\theta = -ma$$
$$a = -g\sin\theta$$

In our assumption ⍰ is very small because displacement is small, in this condition we can take $\sin\theta = \theta$
Hence

$$a = -g\theta \qquad (1)$$

If x be the linear displacement of the bob from its mean position, then from figure, the length of arc AB is nearly equal to x. From elementary geometry we know that:

$$x = l \cdot \theta$$

$$\theta = \frac{x}{l}$$

Putting the value of θ in equation (1)

$$a = -g\frac{x}{l}$$

For a given pendulum g and length are constant:

$$a = -(constant) \cdot x$$

$$a = \ddot{x} \sim x$$

As the acceleration of the bob of simple pendulum is directly proportional to displacement and is directed towards the mean position, therefore the motion is simple harmonic when it is given a small displacement.

#2

Formula for the period:

$$T = 2\pi\sqrt{\frac{l}{g}}; \quad (2) \ \ \theta \ll 1$$

For larger amplitudes, the period increases gradually with amplitude so it is longer than given by equation (2). For example, at an amplitude of θ = 23° it is 1% larger than given by (2).

The period increases asymptotically (to infinity) as θ approaches 180°, because the value θ = 180° is an unstable equilibrium point for the pendulum. The true period of an ideal simple gravity pendulum can be written in several different forms one example being the infinite series:

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\,\theta^2 + \frac{11}{3072}\theta^4 \ldots\right)$$

Hence, time period of simple pendulum of very large length is independent of length.