Task. A water fountain ejects water at a 73 degree angle from a height of 10 cm and lands 20 cm away with a final height of 0 cm. What is the initial speed of the water? How high does the water go?

Solution. Let v_0 be the initial velocity of the water and $\phi = 73°$ be the angle of the initial velocity with horizontal axis. Then its motion can be decomposed as a sum of vertical and horizontal motion. The vertical motion has constant acceleration $g = 9.88\ m/s^2$ due to gravity and initial velocity $v_y^0 = v_0 \sin\phi$, while the horizontal motion has constant velocity $v_x^0 = v_0 \cos\phi$.

By assumption the water passes distance $d_1 = 20\ cm$ away from the initial horizontal position before landing the ground, therefore the time of water motion is

$$t_1 = \frac{d_1}{v_x^0} = \frac{d_1}{v_0 \cos\phi}.$$

Let $h_0 = 10\ cm$ be the initial height of the water. Then the height at time t is given by the formula:

$$h(t) = h_0 + v_y^0 t - \frac{gt^2}{2},$$

and we by assumption

$$h(t_1) = 0.$$

Substituting t_1 we get:

$$0 = h_1(t_1) = h_0 + v_y^0 t_1 - \frac{gt_1^2}{2} = h_0 + v_0 \sin\phi \cdot \frac{d_1}{v_0 \cos\phi} - \frac{g \cdot \left(\frac{d_1}{v_0 \cos\phi}\right)^2}{2}$$
$$= h_0 + d_1 \tan\phi - \frac{gd_1^2}{2v_0^2 \cos^2\phi}$$

Hence

$$v_0 = \sqrt{\frac{gd_1^2}{2(h_0 + d_1 \tan\phi)\cos^2\phi}} = \frac{d_1}{\cos\phi} \cdot \sqrt{\frac{g}{2(h_0 + d_1 \tan\phi)}}$$

Substituting values we get:

$$v_0 = \frac{d_1}{\cos\phi} \cdot \sqrt{\frac{g}{2(h_0 + d_1 \tan\phi)}} = \frac{0.2}{\cos 73°} \cdot \sqrt{\frac{9.81}{2 \cdot (0.1 + 0.2 \cdot \tan 73°)}} \approx 1.74\ m/s.$$

Notice that the vertical velocity of the water is given by the formula:

$$v_y(t) = v_y^0 - gt.$$

The maximal height of water is achieved at time t when $v_y(t) = 0$:

$$v_y(t) = v_y^0 - gt = 0, \qquad \Rightarrow \qquad t = \frac{v_y^0}{g} = \frac{v_0 \sin\phi}{g}.$$

Substituting this t into the formula for $h(t)$ we get the maximal values of height:

$$h(t) = h_0 + v_0 \sin\phi \cdot \frac{v_0 \sin\phi}{g} - \frac{g\left(\frac{v_0 \sin\phi}{g}\right)^2}{2}$$
$$= h_0 + \frac{(v_0 \sin\phi)^2}{g} - \frac{(v_0 \sin\phi)^2}{2g} = h_0 + \frac{(v_0 \sin\phi)^2}{2g}$$

Hence

$$\max(h) = h_0 + \frac{(v_0 \sin\phi)^2}{2g} = 0.1 + \frac{(1.74 \cdot \sin 73°)}{2 \cdot 9.81} \approx 24cm.$$

Answer. $v_0 = 1.74\ m/s, \quad \max h = 24\ cm.$