Question 31703

1. First, let us consider the case when the initial velocity is directed downwards. Let v_0 denote the latter one. Hence, if vertical axis is directed upwards, the law of motion will be $h - v_0 t_1 - \frac{g t_1^2}{2} = 0$ (knowing that t_1 is given time).

2. Secondly, let us consider the case when the initial velocity is directed upwards. First, the body will move upwards until its velocity becomes zero. Let the time needed for this be t' and the height above initial height be h_1. Then, for full stop $v = v_0 - g t' = 0 \Rightarrow t' = \frac{v_0}{g}$, and height h_1 is $h_1 = v_0 t' - g \frac{t'^2}{2} = \frac{v_0^2}{2g}$ (using previous formula for t'). After reaching maximum height ($h + h_1$) and stop, body will move downwards with no initial velocity for some time t''. The latter one is $t'' = \sqrt{2 \frac{(h + h_1)}{g}} = \sqrt{\frac{2}{g}\left(h + \frac{v_0^2}{2g}\right)}$.

Full time of motion in case of initial velocity directed upwards is $t_2 = t' + t'' = \frac{v_0}{g} + \sqrt{\frac{2}{g}\left(h + \frac{v_0^2}{2g}\right)}$.

In case of motion from height h with no initial velocity, time is $t = \sqrt{2 \frac{h}{g}}$.

From points 1 and 2 one has following equations:

a) $h - v_0 t_1 - \frac{g t_1^2}{2} = 0$

b) $t_2 = t' + t'' = \frac{v_0}{g} + \sqrt{\frac{2}{g}\left(h + \frac{v_0^2}{2g}\right)}$

Let us solve b) for v_0 and plug it into a).

$\frac{2}{g}\left(h + \frac{v_0^2}{2g}\right) = t_2^2 - 2 t_2 \frac{v_0}{g} + \frac{v_0^2}{g^2}$ from where $v_0 = \frac{g}{2 t_2}\left(t_2^2 - \frac{2h}{g}\right)$.

Now, plug latter formula for v_0 into a):

$h - \frac{g}{2 t_2}\left(t_2^2 - \frac{2h}{g}\right) t_1 - \frac{g t_1^2}{2} = 0$, from here $h\left(\frac{t_1 + t_2}{t_2}\right) = \frac{g t_1^2}{2} + \frac{g t_1 t_2}{2} = \frac{g}{2}(t_1^2 + t_1 t_2)$.

Hence, $h = \frac{g}{2}\left(\frac{t_2}{t_1 + t_2}\right)(t_1^2 + t_1 t_2) = \frac{g}{2} \frac{(t_2 t_1^2 + t_1 t_2^2)}{t_1 + t_2} = \frac{g}{2} \frac{t_1 t_2 (t_1 + t_2)}{t_1 + t_2} = \frac{g}{2}(t_1 t_2)$.

Finally, knowing that $t = \sqrt{2 \frac{h}{g}}$, obtain $t = \sqrt{t_1 t_2}$.