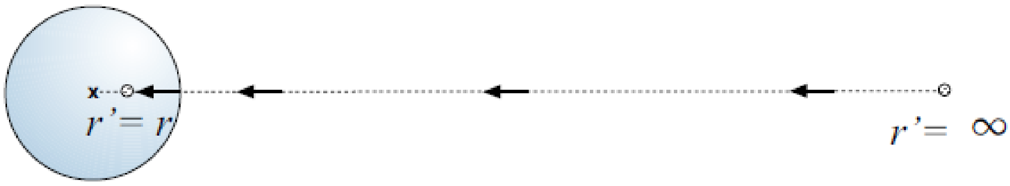
Question:

A charge q is distributed uniformly throughout a spherical volume of radius R.

Setting V=0 at infinity, show that the potential at a distance r from the center, where r<R, is given by

$V = q\frac{[3R^2 - r^2]}{8\pi\epsilon R^3}$

Solution:



Path of integration. Integration goes from $r' = \infty$ to $r' = r$.

This time the reference point (the place where V = 0) is at $r = \infty$. So we will evaluate V (r):

$$V(\boldsymbol{r}) = -\int_{r_{ref}}^{r} \boldsymbol{E}\cdot d\boldsymbol{s} = -\int_{\infty}^{r} E_r(r')\cdot dr'. \ (1)$$

The integration path is shown on the picture above.

We note that the integration (from $r' = \infty$ *to* $r' = r$ with r < R) is over values of r both outside and inside the sphere.

The $\boldsymbol{E}$ field for points inside the sphere is

$$E_{r,in}(r) = \frac{qr}{4\pi\varepsilon_0 R^3}, \ (2)$$

but we will also need the value of the E field outside the sphere.

By Gauss'(s) law the external E field is that same as that due to a point charge q at distance r, so:

$$E_{r,\ out}(r) = \frac{q}{4\pi\varepsilon_0 r^2}. \ (3)$$

Because $E_r(r)$ has two different forms for the interior and exterior of the sphere, we will have to split up the integral in (1) into two parts. When we go from ∞ to R we need to use (3) for $E_r(r')$. When we go from R to r we need to use (2) for $E_r(r')$.

So from (1) we now have

$$V(\boldsymbol{r}) = -\int_{\infty}^{R} E_{r,\ out}(r')\cdot dr' - \int_{R}^{r} E_{r,\ in}(r')\cdot dr' = -\int_{\infty}^{R}\left(\frac{q}{4\pi\varepsilon_0 r'^2}\right)dr' - \int_{R}^{r}\left(\frac{qr'}{4\pi\varepsilon_0 R^3}\right)dr'$$

$$= -\frac{q}{4\pi\varepsilon_0}\left\{\int_{\infty}^{R}\left(\frac{dr'}{r'^2}\right) + \int_{R}^{r}\left(\frac{r'}{R^3}\right)dr'\right\}$$

Now do the individual integrals and we're done:

$$V(\boldsymbol{r}) = -\frac{q}{4\pi\varepsilon_0}\left\{-\frac{1}{r'}\Big|_{\infty}^{R} + \frac{r'^2}{2R^3}\Big|_{R}^{r}\right\} = -\frac{q}{4\pi\varepsilon_0}\left\{-\frac{1}{R} + \frac{r^2 - R^2}{2R^3}\right\} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{2R^2}{2R^3} + \frac{R^2 - r^2}{2R^3}\right\}$$

$$= \frac{q(R^2 - r^2)}{8\pi\varepsilon_0 R^3}$$