How can I multiply the initial density matrix by the time evolution (U) to get the density matrix at any time ?

Answer

Let's discuss the time evolution of mixed states. In the case of a bipartite pure state governed by the usual axioms of quantum theory, let us suppose that the Hamiltonian on $H_A \otimes H_B$ has the form

$$H_{AB} = H_A \otimes 1_B + 1_A \otimes H_B.$$

Under this assumption, there is no coupling between the two subsystems A and B, so that each evolves independently. The time evolution operator for the combined system

$$U_{AB}(t) = U_A(t) \otimes U_B(t),$$

decomposes into separate unitary time evolution operators acting on each system.

In the Schrodinger picture of dynamics, then, an initial pure state $|\psi(0)\rangle_{AB}$ of the bipartite system given by eq. (2.57) evolves to

$$|\psi(0)\rangle_{AB} = \sum_{i,\mu} ai\mu \, |i(t)\rangle_A \otimes |\mu(t)\rangle_B, \text{(2.77)}$$

where

$$|i(t)\rangle_A = U_A(t)|i(0)\rangle_A,$$

$$|\mu(t)\rangle_B = U_B(t)|\mu(0)\rangle_B, \text{(2.78)}$$

define new orthonormal basis for H_A and H_B (since $U_A(t)$ and $U_B(t)$ are unitary). Taking the partial trace as before, we find

$$\rho_A(t) = \sum_{i,j,\mu} a_{i,\mu}\, a^*_{j\nu} \, |i(t)\rangle_{A\,A}\langle j(t)| = U_A(t)\rho_A(0)U_A(t)^\dagger.$$

Thus $U_A(t)$, acting by conjugation, determines the time evolution of the density matrix.

In particular, in the basis in which $\rho_A(0)$ is diagonal, we have

$$\rho_A(t) = \sum_a p_a U_A(t)|\psi(0)\rangle_{A\,A}\langle\psi_a(0)|U_A(t). \text{(2.80)}$$

Eq. (2.80) tells us that the evolution of ρ_A is perfectly consistent with the ensemble interpretation. Each state in the ensemble evolves forward in time governed by $U_A(t)$. If the state $|\psi_a(0)\rangle$ occurs with probability p_a at time 0,then $|\psi_a(0)\rangle$ occurs with probability pa at the subsequent time t.

On the other hand, it should be clear that eq. (2.80) applies only under the assumption that systems A and B are not coupled by the Hamiltonian. Later, we will investigate how the density matrix evolves under more general conditions.