Answer to Question #90371 – Math – Linear Algebra

Question

Let A be a matrix of 3×2 order with real entities. H=A(A^TA)^-1A^T, where A^T is the transpose of the matrix. Let I be the identity matrix of the order 3×3. H^2=?

Solution

A be a 3×2 matrix. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

Given that $H = A_{3\times2}({A^T}_{2\times3}A_{3\times2})^{-1}{A^T}_{2\times3}$

$= A_{3\times2}(B)^{-1}{}_{2\times2}{A^T}_{2\times3}$ (Since the order of $B = {A^T}_{2\times3}A_{3\times2}$ is 2×2)

Consider, $H^2 = \left(A_{3\times2}(B)^{-1}{}_{2\times2}{A^T}_{2\times3}\right)^2$

$= \left(A_{3\times2}(B)^{-1}{}_{2\times2}{A^T}_{2\times3}\right)\left(A_{3\times2}(B)^{-1}{}_{2\times2}{A^T}_{2\times3}\right)$

$= A_{3\times2}(B)^{-1}{}_{2\times2}({A^T}_{2\times3}A_{3\times2})(B)^{-1}{}_{2\times2}{A^T}_{2\times3}$

(Since Matrix multiplication is associative)

$= A_{3\times2}(B)^{-1}{}_{2\times2}(B_{2\times2})(B)^{-1}{}_{2\times2}{A^T}_{2\times3}$ (Since, $B_{2\times2} = {A^T}_{2\times3}A_{3\times2}$)

$= A_{3\times2}(B)^{-1}{}_{2\times2}(I_{2\times2}){A^T}_{2\times3}$ (Since, $(B_{2\times2})(B)^{-1}{}_{2\times2} = I_{2\times2}$)

$= A_{3\times2}(B)^{-1}{}_{2\times2}{A^T}_{2\times3}$ (Since, $I_{2\times2}{A^T}_{2\times3} = {A^T}_{2\times3}$)

$= A_{3\times2}C_{2\times3}$(Since, $(B)^{-1}{}_{2\times2}{A^T}_{2\times3} = C_{2\times3}$)

$=D_{3\times3}$

Therefore, $H^2 = \left(A_{3\times2}(B)^{-1}{}_{2\times2}{A^T}_{2\times3}\right)^2 = D_{3\times3}$.

In this problem we cannot apply $(AB)^{-1} = (B)^{-1}(A)^{-1}$ (Since, A and Transpose A are not invertible in the given problem as they are rectangular matrices)
