$$y''' + 4y' = 3x - 1$$

This is linear non-homogeneous ordinary differential equation with constant coefficients. For this type of differential equation, the general solution is the superposition of the particular solution $y_p(x)$ of the homogeneous equation and the complementary solution $y_c(x)$

We seek the solution of the homogeneous equation:

$$y''' + 4y' = 0$$

$$k^3 + 4k = 0$$

$$k(k^2 + 4) = 0$$

$$k_1 = 0, \qquad k_2 = 2i, \quad k_3 = -2i$$

Solution of the homogeneous equation:

$$y_p(x) = C_1 + C_2\cos 2x + C_3\sin 2x$$

The complementary solution $y_c(x)$ in will be sought in the form

$$y_c(x) = x * (Ax + B)$$

Find y''' and y' we will have:

$$8Ax + 4B = 3x - 1$$

So $A = \frac{3}{8}$ and $B = \frac{-1}{4}$

$$y_c(x) = \frac{3}{8}x^2 - \frac{1}{4}x$$

The general solution is

$$y = C_1 + C_2\cos 2x + C_3\sin 2x + \frac{3}{8}x^2 - \frac{1}{4}x$$

Answer:

$y = C_1 + C_2\cos 2x + C_3\sin 2x + \frac{3}{8}x^2 - \frac{1}{4}x$