Solve $9 \frac{d^2 y(x)}{dx^2} + y(x) = (x^2 + 1) \sin(3 x)$:

The general solution will be the sum of
the complementary solution and particular solution.

Find the complementary solution by solving $9 \frac{d^2 y(x)}{dx^2} + y(x) = 0$:

Assume a solution will be proportional to $e^{\lambda x}$ for some constant λ.

Substitute $y(x) = e^{\lambda x}$ into the differential equation:

$$9 \frac{d^2}{dx^2}(e^{\lambda x}) + e^{\lambda x} = 0$$

Substitute $\frac{d^2}{dx^2}(e^{\lambda x}) = \lambda^2 e^{\lambda x}$:

$$9 \lambda^2 e^{\lambda x} + e^{\lambda x} = 0$$

Factor out $e^{\lambda x}$:

$$(9 \lambda^2 + 1) e^{\lambda x} = 0$$

Since $e^{\lambda x} \neq 0$ for any finite λ, the zeros must come from the polynomial:

$$9 \lambda^2 + 1 = 0$$

Solve for λ:

$\lambda = \frac{i}{3}$ or $\lambda = -\frac{i}{3}$

The roots $\lambda = \pm \frac{i}{3}$ give $y_1(x) = c_1 e^{\frac{i x}{3}}$, $y_2(x) = c_2 e^{-\frac{i x}{3}}$
as solutions, where c_1 and c_2 are arbitrary constants.

The general solution is the sum of the above solutions:

$$y(x) = y_1(x) + y_2(x) = c_1 e^{\frac{i x}{3}} + c_2 e^{-\frac{i x}{3}}$$

Apply Euler's identity $e^{\alpha + i \beta} = e^{\alpha} \cos(\beta) + i e^{\alpha} \sin(\beta)$:

$$y(x) = c_1 \left(\cos\left(\frac{x}{3}\right) + i \sin\left(\frac{x}{3}\right)\right) + c_2 \left(\cos\left(\frac{x}{3}\right) - i \sin\left(\frac{x}{3}\right)\right)$$

Regroup terms:

$$y(x) = (c_1 + c_2) \cos\left(\frac{x}{3}\right) + i (c_1 - c_2) \sin\left(\frac{x}{3}\right)$$

Redefine $c_1 + c_2$ as c_1 and $i (c_1 - c_2)$ as c_2, since these are arbitrary constants:

$$y(x) = c_1 \cos\left(\frac{x}{3}\right) + c_2 \sin\left(\frac{x}{3}\right)$$

Determine the particular solution to $9 \frac{d^2 y(x)}{dx^2} + y(x) = \left(x^2 + 1\right) \sin(3 x)$ by the method of undetermined coefficients:

The particular solution to $9 \frac{d^2 y(x)}{dx^2} + y(x) = \left(x^2 + 1\right) \sin(3 x)$ is of the form:

$$y_p(x) = a_1 \cos(3 x) + a_2 x \cos(3 x) + a_3 x^2 \cos(3 x) + a_4 \sin(3 x) + a_5 x \sin(3 x) + a_6 x^2 \sin(3 x)$$

Solve for the unknown constants a_1, a_2, a_3, a_4, a_5, and a_6:

Compute $\frac{d^2 y_p(x)}{dx^2}$:

$$\begin{aligned}\frac{d^2 y_p(x)}{dx^2} &= \frac{d^2}{dx^2}\left(a_1 \cos(3 x) + a_2 x \cos(3 x) + a_3 x^2 \cos(3 x) + a_4 \sin(3 x) + a_5 x \sin(3 x) + a_6 x^2 \sin(3 x)\right) \\ &= -9 a_1 \cos(3 x) - 9 a_2 x \cos(3 x) - 6 a_2 \sin(3 x) + 2 a_3 \cos(3 x) - 9 a_3 x^2 \cos(3 x) - 12 a_3 x \sin(3 x) - 9 a_4 \sin(3 x) + 6 a_5 \cos(3 x) - \\ &\quad 9 a_5 x \sin(3 x) + 12 a_6 x \cos(3 x) + 2 a_6 \sin(3 x) - 9 a_6 x^2 \sin(3 x)\end{aligned}$$

Substitute the particular solution $y_p(x)$ into the differential equation:

$$9\frac{d^2 y_p(x)}{dx^2} + y_p(x) = \sin(3x) + x^2 \sin(3x)$$

$$9\big(-9 a_1 \cos(3x) - 9 a_2 x \cos(3x) - 6 a_2 \sin(3x) + 2 a_3 \cos(3x) - 9 a_3 x^2 \cos(3x) - 12 a_3 x \sin(3x) - 9 a_4 \sin(3x) + 6 a_5 \cos(3x) - 9 a_5 x \sin(3x) + 12 a_6 x \cos(3x) + 2 a_6 \sin(3x) - 9 a_6 x^2 \sin(3x)\big) + \big(a_1 \cos(3x) + a_2 x \cos(3x) + a_3 x^2 \cos(3x) + a_4 \sin(3x) + a_5 x \sin(3x) + a_6 x^2 \sin(3x)\big) = \sin(3x) + x^2 \sin(3x)$$

Simplify:

$$(-80 a_1 + 18 a_3 + 54 a_5) \cos(3x) + (-80 a_2 + 108 a_6) x \cos(3x) - 80 a_3 x^2 \cos(3x) + (-54 a_2 - 80 a_4 + 18 a_6) \sin(3x) + (-108 a_3 - 80 a_5) x \sin(3x) - 80 a_6 x^2 \sin(3x) = \sin(3x) + x^2 \sin(3x)$$

Equate the coefficients of $\cos(3x)$ on both sides of the equation:

$$-80 a_1 + 18 a_3 + 54 a_5 = 0$$

Equate the coefficients of $x \cos(3x)$ on both sides of the equation:

$$-80 a_2 + 108 a_6 = 0$$

Equate the coefficients of $x^2 \cos(3x)$ on both sides of the equation:

$$-80 a_3 = 0$$

Equate the coefficients of $\sin(3x)$ on both sides of the equation:

$$-54 a_2 - 80 a_4 + 18 a_6 = 1$$

Equate the coefficients of $x \sin(3x)$ on both sides of the equation:

$$-108 a_3 - 80 a_5 = 0$$

Equate the coefficients of $x^2 \sin(3x)$ on both sides of the equation:

$$-80\, a_6 = 1$$

Solve the system:

$$a_1 = 0$$
$$a_2 = -\frac{27}{1600}$$
$$a_3 = 0$$
$$a_4 = -\frac{251}{64000}$$
$$a_5 = 0$$
$$a_6 = -\frac{1}{80}$$

Substitute a_1, a_2, a_3, a_4, a_5, and a_6 into

$$y_p(x) = a_1 \cos(3x) + a_2\, x \cos(3x) + a_3\, x^2 \cos(3x) + a_4 \sin(3x) + a_5\, x \sin(3x) + a_6\, x^2 \sin(3x):$$

$$y_p(x) = -\frac{27\, x \cos(3x)}{1600} - \frac{251 \sin(3x)}{64000} - \frac{1}{80}\, x^2 \sin(3x)$$

The general solution is:

$$y(x) = y_c(x) + y_p(x) = -\frac{27\, x \cos(3x)}{1600} - \frac{251 \sin(3x)}{64000} - \frac{1}{80}\, x^2 \sin(3x) + c_1 \cos\left(\frac{x}{3}\right) + c_2 \sin\left(\frac{x}{3}\right)$$