Answer to Question #88644 – Math – Trigonometry

Question

Find all the complex roots. Leave your answers in polar form with the argument in degrees.

14) The complex fourth roots of -16

Solution

The generalized expression of a complex number in polar coordinate is written as:

$$z = re^{i\theta} = r \times (\cos\theta + i\sin\theta) \text{ Where } \theta = \arg z$$

Now, we can rewrite the above complex number expression into more generalized polar form of complex number as:

$$z = r \times e^{(i\theta + 2n\pi i)} = r \times e^{\{i \times (\theta + 2n\pi)\}} = r \times e^{\{i \times (\theta + n \times 360^\circ)\}}, \text{ Where } n = 0, \pm 1, \pm 2, ...$$

Now, we have to obtain complex fourth root of -16. So, mathematically, we have to obtain $(-16)^{\frac{1}{4}}$. Let us assume that: $z = (-16)^{\frac{1}{4}} \Rightarrow z^4 = -16$

So, we first express this expression $z^4 = -16$, into polar coordinate. This is as below:

$$z^4 = -16$$
$$\Rightarrow z^4 = 16 \times -1$$
$$\Rightarrow z^4 = 16 \times \{1 \times (\cos 180^\circ + \sin 180^\circ)\}$$
$$\Rightarrow z^4 = 16 \times e^{(i \times 180^\circ)}$$

So, we can rewrite this expression into more generalized polar form of complex number as:

$$z^4 = 16 \times e^{(i \times 180^\circ)}$$
$$\Rightarrow z^4 = 16 \times e^{i(180^\circ + n \times 360^\circ)} \quad \text{Where } n = 0, \pm 1, \pm 2, ...$$

Now, De Moivre's Theorem to obtain the *k*-th root of a complex number, is written as:

$$\left(e^{i\theta}\right)^{1/k} = (\cos\theta + i\sin\theta)^{1/k} = \left[\cos\left\{\frac{(\theta + n \times 360^\circ)}{k}\right\} + \sin\left\{\frac{(\theta + n \times 360^\circ)}{k}\right\}\right]$$

Where $n = 0, \pm 1, \pm 2, ...$

So, using this De Moivre's Theorem, the 4-th root of the given complex number becomes:

$$z^4 = 16 \times e^{i\left(180^\circ + n\times 360^\circ\right)}$$

$$\Rightarrow z = (16)^{\frac{1}{4}} \times e^{\left(\left\{i\times\left(180^\circ + n\times 360^\circ\right)\right\}/4\right)}$$

$$\Rightarrow z = 2\times e^{\left(\left\{i\times\left(180^\circ + n\times 360^\circ\right)\right\}/4\right)}$$

So, now putting the values of $n = 0, \pm 1, \pm 2, ...$, we can get individual roots of the given number. So, rewriting the expression again, we get:

$$(-16)^{\frac{1}{4}} = 2\times e^{\left(\left\{i\times\left(180^\circ + n\times 360^\circ\right)\right\}/4\right)} \quad \text{Where } n = 0, \pm 1, \pm 2, ...$$

So, we can write the general expression of root for this complex number as:

$$z = (-16)^{\frac{1}{4}} = 2\times e^{\left(\left\{i\times\left(180^\circ + n\times 360^\circ\right)\right\}/4\right)} \quad \text{Where } n = 0, \pm 1, \pm 2, ...$$

Now, to get the distinct roots, we have to put the values of:

$$n = 0, 1, 2, 3$$

And accordingly, the individual root becomes:

First Root (z_1) for *n = 0:*

$$z_1 = 2\times e^{\left(\left\{i\times\left(180^\circ + 0\times 360^\circ\right)\right\}/4\right)} = 2e^{i\times 45^\circ} = 2\left(\cos 45^\circ + i\sin 45^\circ\right)$$

Second Root (z_2) for *n = 1:*

$$z_2 = 2\times e^{\left(\left\{i\times\left(180^\circ + 1\times 360^\circ\right)\right\}/4\right)}$$

$$\Rightarrow z_2 = 2e^{i\times\left(\frac{540^\circ}{4}\right)}$$

$$\Rightarrow z_2 = 2e^{i\times 135^\circ}$$

$$\Rightarrow z_2 = 2\left(\cos 135^\circ + i\sin 135^\circ\right)$$

$$\Rightarrow z_2 = 2\left\{\cos\left(90+45\right)^\circ + i\sin\left(90+45\right)^\circ\right\}$$

$$\Rightarrow z_2 = 2\left(-\sin 45^\circ + i\cos 45^\circ\right)$$

Third Root (z_3) for *n = 2:*

$$z_3 = 2 \times e^{\left(\left\{i\times\left(180^\circ + 2\times 360^\circ\right)\right\}/4\right)}$$

$$\Rightarrow z_3 = 2e^{i\times\left(\frac{900^\circ}{4}\right)}$$

$$\Rightarrow z_3 = 2e^{i\times 225^\circ}$$

$$\Rightarrow z_3 = 2\left(\cos 225^\circ + i\sin 225^\circ\right)$$

$$\Rightarrow z_3 = 2\left\{\cos\left(180+45\right)^\circ + i\sin\left(180+45\right)^\circ\right\}$$

$$\Rightarrow z_3 = 2\left(-\cos 45^\circ - i\sin 45^\circ\right)$$

And finally the fourth Root (z_4) for *$n = 3$:*

$$z_4 = 2 \times e^{\left(\left\{i\times\left(180^\circ + 3\times 360^\circ\right)\right\}/4\right)}$$

$$\Rightarrow z_4 = 2e^{i\times\left(\frac{1260^\circ}{4}\right)}$$

$$\Rightarrow z_4 = 2e^{i\times 315^\circ}$$

$$\Rightarrow z_4 = 2\left(\cos 315^\circ + i\sin 315^\circ\right)$$

$$\Rightarrow z_4 = 2\left\{\cos\left(360-45\right)^\circ + i\sin\left(360-45\right)^\circ\right\}$$

$$\Rightarrow z_4 = 2\left(\cos 45^\circ - i\sin 45^\circ\right)$$