Answer to Question #87224 – Math – Abstract Algebra

Question

a) If G is a group of order 40, and H and K are its subgroups of orders 20 and 10, then check whether or not $HK \leq G$. Further, show that $o(H \cap K) \geq 5$.

Proof. Since $[G:H] = 2$, H is a normal subgroup of G and hence $HK \leq G$. Since $o(H) = 20$ and $o(K) = 10$, we have $o(HK) = o(H)*o(K)/o(H \cap K) = 200/o(H \cap K)$. If $o(H \cap K) < 5$, then $o(HK) > 40 = o(G)$, which is not possible. Therefore, $o(H \cap K) \geq 5$.

Question

b) Prove that $C^*/S \cong R^+$, where $S = \{z \in C^* \mid |z|=1\}$, $R^+ = \{x \in R \mid x>0\}$ and $C^* = C \backslash \{0\}$.

Proof. Let $f: C^* \to R^+$ be defined by $f(z) = |z|$. Let $z_1, z_2 \in C^*$. Then, $f(z_1 z_2) = |z_1 z_2| = |z_1||z_2| = f(z_1) f(z_2)$. Therefore, f is a homomorphism. Let $S = \{z \in C^* \mid |z|=1\}$. Then $z_1 = z_2$ in C^*/S if and only if $z_1 / z_2 \in S$ if and only if $|z_1 / z_2| = 1$ if and only if $|z_1| = |z_2|$. That is $z_1 = z_2$ in C^*/S exactly when they have the same modulus. Thus, every element of G is equal in C^*/S to precisely one nonzero real number. Also, Kernel f = $\{z \in C^*: f(z) = 1\} = \{z \in C^* \mid |z|=1\} = S$. Thus, by First theorem of Homomorphism, $C^*/S \cong R^+$.

Question

c) What are the possible algebraic structures of a group of order 99?

Solution. Let G be a group of order 99. From the Sylow's theorem, there is only one 3-Sylow subgroup and only one 11-Sylow subgroup. Let A be a 3-Sylow subgroup, B is an 11-Sylow subgroup. Hence both A and B are normal in G. Also, $A \cap B = \{e\}$ and since $|AB|=|A||B|/|A \cap B|= (9 \times 11) / 1 = 99 = |G|$, we conclude that G = AB. Thus, $G \cong A \times B$. Now, the only group of order 11 up to isomorphism is Z_{11}, and the only groups of order 9 are Z_9 and $Z_3 \times Z_3$. Hence we get that $G \cong Z_9 \times Z_{11}$ or $G = Z_3 \times Z_3 \times Z_{11}$.

Note : Solution for (c) uses the help from https://math.stackexchange.com/questions/3123910/how-do-you-deduce-the-possible-algebraic-structures-of-a-group-from-its-order/3123918.