Answer on Question #85698 – Math – Real Analysis

Question

Obtain the value of x for which the series

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (4n-3)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (4n-2)} \cdot \frac{x^{2n}}{4n}$$

is convergent.

Solution

Use the Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (4n-3)(4(n+1)-3)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (4n-2)(4(n+1)-2)} \cdot \frac{x^{2(n+1)}}{4(n+1)}}{\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (4n-3)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (4n-2)} \cdot \frac{x^{2n}}{4n}} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{n(4n+1)x^2}{(n+1)(4n+2)} \right|$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{n(4n+1)x^2}{(n+1)(4n+2)} \right| \to x^2 \text{ as } n \to \infty$$

By the Ratio Test, the given series converges is $x^2 < 1$ and diverges if $x^2 > 1$.
If $x^2 = 1$, we have the series

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (4n-3)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (4n-2)} \cdot \frac{1}{4n} = \sum_{n=1}^{\infty} \frac{(4n-3)!!}{(4n)!!}$$

The generating function for the Central Binomial Coefficients is

$$(1-4x)^{-1/2} = \sum_{k=0}^{\infty} \binom{2k}{k} x^k = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + 252x^5 + \cdots$$

We can plug $x = -1/4$

$$\left(1 - 4\left(-\frac{1}{4}\right)\right)^{-1/2} = \sum_{k=0}^{\infty} \binom{2k}{k} \left(-\frac{1}{4}\right)^k = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{4^k\left((k)!\right)^2} =$$

$$= 1 + 2\left(-\frac{1}{4}\right) + 6\left(-\frac{1}{4}\right)^2 + 20\left(-\frac{1}{4}\right)^3 + 70\left(-\frac{1}{4}\right)^4 + 252\left(-\frac{1}{4}\right)^5 + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!}$$

$$\frac{\sqrt{2}}{2} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!}$$

$$1 - \frac{\sqrt{2}}{2} = -\sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!}$$

$$=\frac{1}{2}-\frac{1\cdot 3}{2\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}-\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}+\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}-\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}+\cdots$$

$$=\frac{1}{2\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}+\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}+\cdots=\sum_{n=1}^{\infty}\frac{(4n-3)!!}{(4n)!!}$$

The series

$$\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot \ldots\cdot (4n-3)}{2\cdot 4\cdot 6\cdot \ldots\cdot (4n-2)}\cdot\frac{x^{2n}}{4n}$$

is convergent for $x\in[-1,1]$.