Answer on Question #85330 – Math – Complex Analysis

Question

Using the method of residues, evaluate the following integral:

$$\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta}.$$

Solution

According to the main theorem of residues

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{z=z_k} f(z), \qquad z \in D.$$

where C is the contour of area D, z_k are singular points.

Use the substitution $z = e^{i\theta}$ and use the Euler formulas:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad i.e\ \ \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right);$$

$$3+2\cos\theta = 3+z+\frac{1}{z} = \frac{z^2+3z+1}{z};$$

$$dz = ie^{i\theta}d\theta \ \rightarrow \ d\theta = \frac{dz}{iz}.$$

Segment $[0; 2\pi]$ changing variables can be thought of as changing $\arg z$ points z belonging to the circle. Indeed, the substitution $z = e^{i\theta}$ translates the segment $[0; 2\pi]$ to the circle $|z| = 1$, $0 \leq \arg z \leq 2\pi$.

We use all the above

$$\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \oint_{|z|=1} \frac{z}{z^2+3z+1}\frac{dz}{iz} = \frac{1}{i}\oint_{|z|=1} \frac{1}{z^2+3z+1}dz$$

The singular points of the integrand are the zeros of the denominator, that is, the roots of the equation $z^2+3z+1=0$. These are points $z_1 = \frac{-3+\sqrt{5}}{2}$ and $z_2 = \frac{-3-\sqrt{5}}{2}$. Then the denominator can be written as $(z-z_1)(z-z_2)$. Point z_2 does not belong to the domain $|z|<1$ and point z_1 belongs and z_1 is a pole of the 1st order. Find the residue at the point $z = z_1$, which is a pole of the first order:

$$\operatorname*{res}_{z=z_1} \frac{1}{(z-z_1)(z-z_2)} = \lim_{z\to z_1} \frac{1}{(z-z_1)(z-z_2)}(z-z_1) = \lim_{z\to z_1} \frac{1}{(z-z_2)} = \frac{1}{z_1-z_2} = \frac{1}{\sqrt{5}}$$

$$\frac{1}{i}\oint_{|z|=1} \frac{1}{z^2+3z+1}dz = \frac{1}{i}\cdot 2\pi i \cdot \operatorname*{res}_{z=z_1} \frac{1}{(z-z_1)(z-z_2)} = \frac{2\pi}{\sqrt{5}}.$$

Answer:

$$\int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta} = \frac{2\pi}{\sqrt{5}}.$$