Answer to Question #85228 - Math – Calculus

Question:

If $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. Then show that $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$.

Solution:

Let $I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$.

Given $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ implies that

$I = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx = \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$, since $\sin(\pi-x) = \sin x$ and $\cos(\pi-x) = -\cos x$.

$$I + I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx + \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}dx$$

$$2I = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}dx$$

Put $u = \cos x$, then $du = -\sin x dx$. When *x* varies from 0 to π, *u* varies from 1 to -1.

$$2I = \int_1^{-1} \frac{\pi}{1+u^2} - du = -\pi\left[\arctan u\right]_1^{-1} = -\pi\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{2}$$

$$I = \frac{\pi^2}{4}$$

Thus $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$.