ANSWER on Question #84877 – Math – Differential Equations

QUESTION

Under certain condition cane sugar in water is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If of $75gm$, at time $t = 0$, $8gm$, are converted during the first 30 minutes, the amount converted in one and half hours is

SOLUTION

Let

$$C - is\ converted\ sugar\ cane \rightarrow \begin{cases} \frac{dC}{dt} is\ conversion\ rate \\ (75 - C)\ is\ unconverted\ sugar\ cane \end{cases}$$

Then, by the condition of the problem

$$\frac{dC}{dt} \sim (75 - C) \rightarrow \frac{dC}{dt} = k(75 - C), \qquad where\ k\ is\ the\ coefficient\ of\ proportionality$$

Then,

$$\frac{dC}{dt} = k(75 - C) \bigg| \times \left(\frac{dt}{75 - C}\right) \rightarrow \frac{dC}{75 - C} = kdt \rightarrow \int \frac{dC}{75 - C} = \int kdt \rightarrow$$

$$-\ln|75 - C| = kt - \ln|A| | \times (-1) \rightarrow \ln|75 - C| = -kt + \ln|A| \rightarrow$$

$$e^{\ln|75 - C|} = e^{-kt + \ln|A|} \rightarrow 75 - C = A \cdot e^{-kt} \rightarrow \boxed{C(t) = 75 - A \cdot e^{-kt}}$$

By the condition of the problem:

$$C(0) = 0 = 75 - A \cdot e^{-k \cdot 0} \rightarrow 75 - A = 0 \rightarrow \boxed{A = 75}$$

$$C(t) = 75 - 75 \cdot e^{-kt} \rightarrow C(t) = 75 \cdot (1 - e^{-kt})$$

$$C(30) = 8 = 75 \cdot (1 - e^{-k \cdot 30}) \rightarrow 1 - e^{-k \cdot 30} = \frac{8}{75} \rightarrow e^{-30k} = 1 - \frac{8}{75} \rightarrow$$

$$e^{-30k} = \frac{75 - 8}{75} \rightarrow \ln|e^{-30k}| = \ln\left|\frac{67}{75}\right| \rightarrow -30k = \ln\left|\frac{67}{75}\right| \rightarrow k = -\frac{1}{30} \cdot \ln\left|\frac{67}{75}\right| \rightarrow$$

$$k = \ln\left|\left(\frac{67}{75}\right)^{-1/30}\right| = \ln\left|\left(\frac{75}{67}\right)^{1/30}\right| \rightarrow \boxed{k = \ln\left|\left(\frac{75}{67}\right)^{1/30}\right|}$$

Then,

$$C(t) = 75 \cdot \left(1 - e^{-t \cdot \ln\left|\left(\frac{75}{67}\right)^{\frac{1}{30}}\right|}\right) = 75 \cdot \left(1 - e^{\ln\left|\left(\frac{75}{67}\right)^{\frac{-t}{30}}\right|}\right) = 75 \cdot \left(1 - \left(\frac{75}{67}\right)^{\frac{-t}{30}}\right) \rightarrow$$

Conclusion,

$$\boxed{C(t) = 75 \cdot \left(1 - \left(\frac{67}{75}\right)^{\frac{t}{30}}\right)}$$

The amount converted in one and half hours is

$$C(90) = 75 \cdot \left(1 - \left(\frac{67}{75}\right)^{\frac{90}{30}}\right) = 75 \cdot \left(1 - \left(\frac{67}{75}\right)^{3}\right) = \frac{121112}{5625} \approx 21.53(gm)$$

ANSWER:

$$C(t) = 75 \cdot \left(1 - \left(\frac{67}{75}\right)^{\frac{t}{30}}\right)$$

$$C(90) = \frac{121112}{5625} \approx 21.53(gm)$$